

# On Scattering of Solitons for Maxwell Equation Coupled to a Particle

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## Abstract

We establish a long time soliton asymptotics for a nonlinear system of Maxwell equation coupled to a charged particle. The coupled system has a six dimensional manifold of soliton solutions. We show that in the large time approximation, any solution, with an initial state close to the solitary manifold, is a sum of a soliton and a dispersive wave which is a solution of the free Maxwell equation. It is assumed that the charge density satisfies the Wiener condition. The proof is based on a development of the general strategy introduced in the papers of Soffer and Weinstein, Buslaev and Perelman, and others: symplectic projection in Hilbert space onto the solitary manifold, modulation equations for the parameters of the projection, and decay of the transversal component.

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<sup>1</sup>Supported partly by DFG grant 436 RUS 113/929/0-1, FWF Project P19138-N13, RFBR grant 07-01-00018a, and RFBR-DFG grant 08-01-91950-NNIOa.

<sup>2</sup>Supported partly by Alexander von Humboldt Research Award.

# 1 Introduction

Our contribution investigates the long time behavior of a single relativistic charge coupled to the Maxwell field. It is convenient to write the equations of motion in Hamiltonian form. The dynamical variables are then the position,  $q$ , of the particle, together with its momentum  $P$ , and the transverse vector potential,  $A$ , together with the canonically conjugate field  $E_s$ , which physically corresponds to transverse electric field. We refer to [37, Chapter 13] for details. In these variables the Hamiltonian function reads

$$H(E_s, A, q, P) = \frac{1}{2} \int \left( |E_s|^2 + |\nabla A|^2 \right) d^3x + \left[ 1 + \left( P - \int \rho(x - q) A(x) d^3x \right)^2 \right]^{1/2}. \quad (1.1)$$

The phase space will be defined below through  $H < \infty$ . The canonical equations of motion follow as

$$\dot{E}_s(x, t) = -\Delta A(x, t) - \Pi_s(\rho(x - q(t))v(t)), \quad \dot{A}(x, t) = -E_s(x, t), \quad (1.2)$$

$$\dot{q}(t) = \frac{P(t) - \int \rho(x - q(t)) A(x, t) d^3x}{\left[ 1 + \left( P(t) - \int \rho(x - q(t)) A(x, t) d^3x \right)^2 \right]^{1/2}} =: v(t), \quad (1.3)$$

$$\dot{P}(t) = \sum_{k=1}^3 v_k(t) \int \rho(x - q(t)) \nabla A_k(x, t) d^3x, \quad (1.4)$$

together with the transversality conditions

$$\nabla \cdot E_s(x, t) = 0, \quad \nabla \cdot A(x, t) = 0. \quad (1.5)$$

Here and below all derivatives are understood in the sense of distributions.  $E_s(x, t)$  is the projection of electric field onto the space of solenoidal (divergence-free) vector fields. In Fourier space the corresponding projector reads

$$\hat{\Pi}_s a = a - \frac{a \cdot k}{k^2} k.$$

$\rho$  is the charge distribution of the particle, on which we will comment below. We use units such that the velocity of light  $c = 1$ ,  $\varepsilon_0 = 1$ , and the mechanical mass of the charge  $m = 1$ .

We note that, because of translational invariance, the total momentum

$$\mathcal{P}(E_s, A, q, P) = P + \int E_s(x) \wedge (\nabla \wedge A(x)) d^3x \quad (1.6)$$

is conserved along sufficiently smooth trajectories of (1.2)-(1.5).

Let us write the system (1.5)-(1.4) as

$$\dot{Y}(t) = F(Y(t)), \quad t \in \mathbb{R}, \quad (1.7)$$

where  $Y(t) := (E_s(x, t), A(x, t), q(t), P(t))$ . Below we always deal with column vectors but often write them as row vectors. The system (1.2)-(1.5) admits special solutions where the charge travels with constant velocity. In analogy with travelling solutions of nonlinear wave equation we call them *solitons*. Explicitly they are given by

$$Y_{a,v}(t) = (E_{s,v}(x - vt - a), A_v(x - vt - a), vt + a, P_v), \quad P_v = v/\sqrt{1 - v^2} + \langle \rho, A_v \rangle \quad (1.8)$$

for all  $a, v \in \mathbb{R}^3$  with  $|v| < 1$ . The states  $S_{a,v} := Y_{a,v}(0)$  form the solitary manifold

$$\mathcal{S} := \{S_{a,v} : a, v \in \mathbb{R}^3, |v| < 1\}. \quad (1.9)$$

For general initial data one expects that for large times the solution splits up into two parts: one piece consists of a soliton with a definite velocity and the second piece are scattered fields escaping to infinity. In fact, this will be our main result. If the initial data are close to the solitary manifold, then we will prove that for large  $t$

$$(E_s(x, t), A(x, t)) \sim (E_{s, v_{\pm}}(x - v_{\pm}t - a_{\pm}), A_{v_{\pm}}(x - v_{\pm}t - a_{\pm})) + W^0(t)\Psi_{\pm}, \quad t \rightarrow \pm\infty. \quad (1.10)$$

Here  $W^0(t)$  is the dynamical group of the free wave equation (Equations (1.2) with  $\rho = 0$  and (1.5)),  $\Psi_{\pm}$  are the corresponding *asymptotic scattered fields*, and the remainder converges to zero *in the global energy norm*, i.e. in the norm of the space  $\mathcal{F} := H_s^0(\mathbb{R}^3) \oplus \dot{H}_s^1(\mathbb{R}^3)$ , see Section 2. For the particle trajectory we prove that

$$\dot{q}(t) \rightarrow v_{\pm}, \quad q(t) \sim v_{\pm}t + a_{\pm}, \quad t \rightarrow \pm\infty. \quad (1.11)$$

The results are established under the following conditions on the charge distribution:  $\rho$  is a real valued function of the Sobolev class  $H^2(\mathbb{R}^3)$ , compactly supported, and spherically symmetric, i.e.

$$\rho, \nabla\rho, \nabla\nabla\rho \in L^2(\mathbb{R}^3), \quad \rho(x) = 0 \text{ for } |x| \geq R_{\rho}, \quad \rho(x) = \rho_1(|x|). \quad (1.12)$$

An essential point of our asymptotic analysis is the Wiener condition

$$\hat{\rho}(k) = (2\pi)^{-3/2} \int e^{ikx} \rho(x) dx \neq 0 \text{ for all } k \in \mathbb{R}^3 \setminus \{0\}. \quad (1.13)$$

The Wiener condition was noted already in the previous works [16, 20, 21, 24, 25, 26]. It expresses that all modes of the Maxwell field are coupled to the particle.

There is no restriction on  $\int |\rho(x)| dx$ . However if  $\int \rho(x) dx \neq 0$ , then the fields have a slow decay as  $|x|^{-2}$  at infinity. With our methods such a decay seems to be difficult to control and we have to impose the charge neutrality condition

$$\int \rho(x) dx = 0. \quad (1.14)$$

(1.14) is a technical condition. Physically one expects (1.10) to hold even without imposing charge neutrality and it is of interest to extend our proof in this direction.

Let us briefly comment on earlier works. The first mathematical investigation is the contribution of Bambusi and Galgani [4]. They consider a non-relativistic kinetic energy for the charge and prove orbital stability of the solitons. The issue was taken up again in [16, 27, 17], where the kinetic energy is taken to be relativistic. In [16] the soliton-type asymptotics for the fields is established, under the Wiener condition, in local energy semi-norms centered at the particle's position  $q(t)$ . The relaxation of acceleration,  $\ddot{q}(t) \rightarrow 0$  as  $t \rightarrow \infty$ , is proved in [27] under the Wiener condition. Scattering behavior of the type (1.10) is established in [17] for weak coupling, i.e.  $\|\rho\|_{L^2} \ll 1$ . The case of a spinning charge is studied in [19].

A long time asymptotics as (1.10) appears also in nonlinear wave equations, like KDV and the  $U(1)$ -invariant nonlinear Schrödinger equation. Of course, in these equations there is no particle degrees of freedom, and  $\mathcal{S}$  corresponds to the solitary wave solutions travelling at constant velocity. In fact our approach relies on and further develops general strategy introduced in [6, 7, 29, 30]. The strategy originates from the techniques of the paper [41] and their developments in [31, 32, 33] in the context of  $U(1)$ -invariant Schrödinger equation. The approach uses the symplectic geometry methods for the Hamilton systems in Hilbert spaces and spectral theory of nonselfadjoint operators.

The invariant manifolds arise automatically for equations with a symmetry Lie group [5, 13, 14]. In particular, our system (1.2)-(1.5) is invariant under translations in  $\mathbb{R}^3$ . The asymptotic stability of the solitary manifold is studied by a linearization of the dynamics (1.7). The linearization will be made on a special curve on the solitary manifold,  $S(t)$ , which is the symplectic orthogonal projection of the solution. Then the linearized equation reads

$$\dot{X}(t) = A(t)X(t), \quad t \in \mathbb{R}, \quad (1.15)$$

where the operator  $A(t)$  corresponds to the linearization at the soliton  $S(t)$ . Furthermore, consider the “frozen” linearized equation (1.15) with  $A(t_1)$  instead of  $A(t)$ . The operator  $A(t_1)$  has zero eigenvalue, and the frozen linearized equation admits linear in  $t$  secular solutions (see (6.24)). The existence of these runaway solutions prohibits the direct application of the Liapunov strategy and is responsible for the instability of the nonlinear dynamics along the manifold  $\mathcal{S}$ . One of crucial observation is that the linearized equation is stable in the *symplectic orthogonal complement* to the tangent space  $\mathcal{T}_{\mathcal{S}}$ . The complement is invariant under the linearized dynamics since the linearized dynamics is Hamiltonian and leaves the symplectic structure invariant.

Our proofs are based on a suitable development of the methods [6, 7, 29, 30]. Let us comment on the main steps.

I. First, we construct the symplectic orthogonal projection  $S(t) = \Pi Y(t)$  of the trajectory  $Y(t)$  onto the solitary manifold  $\mathcal{S}$ . This means that  $S(t) \in \mathcal{S}$ , and the complement vector  $Z(t) := Y(t) - S(t)$  is symplectic orthogonal to the tangent space  $\mathcal{T}_{S(t)}$  for every  $t \in \mathbb{R}$ :

$$Z(t) \perp \mathcal{T}_{S(t)}, \quad t \in \mathbb{R}. \quad (1.16)$$

So, we get the splitting  $Y(t) = S(t) + Z(t)$  and we linearize the dynamics in the *transversal component*  $Z(t)$  along the trajectory.

The *soliton component*  $S(t) = S_{b(t), v(t)}$  satisfies a *modulation equation*. Namely, in the parametrization  $\xi(t) = (c(t), v(t))$  with  $c(t) := b(t) - \int_0^t v(s) ds$ , we have

$$\dot{\xi}(t) = N_1(\xi(t), Z(t)), \quad |N_1(\xi(t), Z(t))| \leq C \|Z(t)\|_{-\beta}^2, \quad (1.17)$$

where  $\|\cdot\|_{-\beta}$  stands for an appropriate weighted Sobolev norm. On the other hand, the transversal component satisfies the *transversal equation*

$$\dot{Z}(t) = A(t)Z(t) + N_2(S(t), Z(t)), \quad (1.18)$$

where  $A(t) = A_{S(t)}$ , and  $N_2(S(t), Z(t))$  is a nonlinear part:

$$\|N_2(S(t), Z(t))\|_{\beta} \leq C \|Z(t)\|_{-\beta}^2, \quad (1.19)$$

where  $\|\cdot\|_{\beta}$  is defined similarly to  $\|\cdot\|_{-\beta}$ . Let us note that the bound (1.19) is not a direct consequence of the linearization since the function  $S(t)$  generally is not a solution of (1.7). The modulation equation and the bound (1.17) play a crucial role in the proof of (1.19).

II. The linearized dynamics (1.15) is nonautonomous. First, let us fix  $t = t_1$  in  $A(t)$  and consider the corresponding “frozen” linear autonomous equation with  $A(t_1)$  instead of  $A(t)$ . We prove the decay

$$\|X(t)\|_{-\beta} \leq \frac{C \|X(0)\|_{\beta}}{(1 + |t|)^{\beta}}, \quad t \in \mathbb{R} \quad (1.20)$$

of the solutions  $X(t)$  to the frozen equation for any  $X(0) \in \mathcal{Z}_{S_1}$  where  $S_1 := S(t_1)$ , and  $\mathcal{Z}_{S_1}$  is the space of vectors  $X$  which are symplectic orthogonal to the tangent space  $\mathcal{T}_{S_1}$ . Let us stress that the decay holds only for the solutions symplectic orthogonal to the tangent space. Basically, the reason of the decay is the fact that the spectrum of the generator  $A(t_1)$  restricted to the space  $\mathcal{Z}_{S_1}$  is purely continuous.

III. We combine the decay (1.20) with the bound (1.17) through the nonlinear equation (1.18). This gives the time decay of the transversal component

$$\|Z(t)\|_{-\beta} \leq \frac{C (\|Z(0)\|_{\beta})}{(1 + |t|)^{\beta}}, \quad t \in \mathbb{R}, \quad (1.21)$$

if the norm  $\|Z(0)\|_\beta$  is sufficiently small. One of the main difficulties in proving the decay (1.21) is the non-autonomous character of the linear part of (1.18). We deduce the decay from the equation (1.18) written in the “frozen” form

$$\dot{Z}(t) = A(t_1)Z(t) + [A(t) - A(t_1)]Z(t) + N_1(S(t), Z(t)), \quad 0 \leq t < t_1, \quad (1.22)$$

with an arbitrary large  $t_1 > 0$ .

IV. The decay (1.21) implies the soliton asymptotics (1.10) and (1.11) by the known techniques of the scattering theory.

**Remarks 1.1** i) The asymptotic stability of the solitary manifold  $\mathcal{S}$  is caused by the *radiation of energy to infinity* which appears as the *local energy decay* (1.21).

ii) The asymptotics (1.10) can be interpreted as the collision of the incident soliton, with a trajectory  $v_-t + a_-$ , with an incident wave  $W_0(t)\Psi_-$ , which results in an outgoing soliton with a new trajectory  $v_+t + a_+$ , and a new outgoing wave  $W_0(t)\Psi_+$ . It suggests to introduce the (nonlinear) *scattering operator*

$$\mathbf{S} : (v_-, a_-, \Psi_-) \mapsto (v_+, a_+, \Psi_+). \quad (1.23)$$

However, a correct definition of the operator is an open problem as well as the question on its *asymptotic completeness* (i.e. on its range).

**Remarks 1.2** i) The strategy [6, 7, 29, 30] was developed also in [8, 9, 10, 11, 28, 34, 35, 36]. Let us stress that all the papers contain several assumptions on the discrete and continuous spectrum of the linearized problem. In our case a complete investigation of the spectrum of the linearized problem is given under the Wiener condition, and there is no need in a priori spectral assumptions.

ii) Note that the Wiener condition is indispensable for our proof of the decay (1.20), and only in one point: in the proof of Lemma 15.2. In all other places we use only the fact that the coupling function  $\rho(x)$  is not identically zero. The other assumptions on  $\rho$  can be weakened: the spherical symmetry is not necessary, and one can assume also that  $\rho$  belongs to a weighted Sobolev space rather than has a compact support.

Our paper is organized as follows. In Section 2, we formulate the main result. In Section 3, we introduce the symplectic projection onto the solitary manifold. The linearized equation is defined in section 4 and studied in the next one. In Section 6, we split the dynamics in two components: along the solitary manifold, and in transversal directions, and we justify the estimate (1.17) concerning the tangential component. The time decay of the transversal component is established in sections 7 - 10 under an assumption on the time decay of the linearized dynamics. In Section 11, we prove the main result. Sections 12 - 18 fill out the gap concerning the time decay of the linearized dynamics. In Appendices A to C we collect some routine calculations.

**Acknowledgments** The authors thank V. Buslaev for numerous lectures on his results and fruitful discussions.

## 2 Main Results

### 2.1 Existence of Dynamics

Let us introduce a phase space for the system (1.5) to (1.4) and state the existence of dynamics. Set  $H^0 = L^2(\mathbb{R}^3, \mathbb{R}^3)$ ,  $\dot{H}^1$  is the closure of  $C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$  with respect to the norm  $\|A\|_1 = |\nabla A| = \|\nabla A\|_{L^2(\mathbb{R}^3, \mathbb{R}^3)}$ . Let  $H_s^0$ ,  $\dot{H}_s^1$  be the subspaces constituted by solenoidal vector fields, namely the closure in  $H^0$ ,  $\dot{H}^1$  respectively of  $C_0^\infty$  vector fields with vanishing divergence. Define the phase space

$$\mathcal{E} = H_s^0 \oplus \dot{H}_s^1 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3,$$

where the norm of  $Y = (E_s, A, q, P)$  is

$$\|Y\|_{\mathcal{E}} = |E_s| + \|A\|_1 + |q| + |P|.$$

Let us define the corresponding space for fields alone:

$$\mathcal{F} = H_s^0 \oplus \dot{H}_s^1, \quad \|(E_s, A)\|_{\mathcal{F}} = |E_s| + \|A\|_1.$$

We write the Cauchy problem for the system (1.5)-(1.4) as

$$\dot{Y} = F(Y(t)), \quad t \in \mathbb{R}; \quad Y(0) = Y^0. \quad (2.1)$$

**Proposition 2.1** *Let (C) holds, let  $Y^0 = (E_s^0, A^0, q^0, P^0) \in \mathcal{E}$ . Then*

*i) There exists a unique solution  $Y(t) \in C(\mathbb{R}, \mathcal{E})$  to the Cauchy problem (2.1).*

*ii) The energy and the total momentum are conserved,*

$$H(Y(t)) = H(Y^0), \quad \mathcal{P}(Y(t)) = \mathcal{P}(Y^0), \quad t \in \mathbb{R}.$$

*iii) The estimate holds,*

$$|\dot{q}(t)| \leq \bar{v} < 1, \quad t \in \mathbb{R}. \quad (2.2)$$

For proof see [16].

## 2.2 Soliton Solutions and the Main Result

We will investigate the long-time behavior of finite energy solutions to (1.5)-(1.4). The set of asymptotic solutions corresponds to the charge travelling with a uniform velocity,  $v$ . Up to translation they are of the form

$$E_s(x, t) = E_{s,v}(x - vt), \quad A(x, t) = A_v(x - vt), \quad q(t) = vt, \quad P_v = p_v + \int \rho A_v d^3x, \quad (2.3)$$

where  $v \in \mathbb{R}^3, |v| < 1$ ;  $E_{s,v} = \Pi_s E^v$ ,  $A_v = \Pi_s A^v$ , and  $E^v, A^v, p_v$  are given by

$$E^v(x) = -\nabla \phi_v(x) + v \cdot \nabla A^v(x), \quad A^v(x) = v \phi_v(x);$$

$$\phi_v(x) = \frac{\gamma}{4\pi} \int \frac{\rho(y) d^3y}{|\gamma(y - x)_{\parallel} + (y - x)_{\perp}|}, \quad p_v = \frac{v}{\sqrt{1 - v^2}}.$$

Here  $\gamma = 1/\sqrt{1 - v^2}$  and  $x = x_{\parallel} + x_{\perp}$ , where  $x_{\parallel} \parallel v$  and  $x_{\perp} \perp v$  for  $x \in \mathbb{R}^3$ . Below we call the solutions of type (2.3) “solitons”. For simplicity of notations let us omit the index  $s$  and write  $E_v$  instead of  $E_{s,v}$ . The stationary equations for  $E_v, A_v, P_v$  read

$$E_v(x) = v \cdot \nabla A_v(x), \quad (2.4)$$

$$v \cdot \nabla E_v(x) = \Delta A_v(x) + \Pi_s(\rho(x)v), \quad (2.5)$$

$$v = \frac{P_v - \int \rho(x) A_v(x) d^3x}{\left[1 + \left(P_v - \int \rho(x) A_v(x) d^3x\right)^2\right]^{1/2}}, \quad (2.6)$$

$$0 = \int \rho(x) v \cdot \nabla A_v(x) d^3x. \quad (2.7)$$

To state our main result we have to introduce the following weighted Sobolev spaces. Let  $H_{s,\alpha}^0, \dot{H}_{s,\alpha}^1$  be the subspaces of  $H_s^0$ , respectively  $\dot{H}_s^1$  consisting of all the fields  $E$ , resp.  $A$  with the finite norms

$$\|E\|_{0,\alpha} = |(1+|x|)^\alpha E|, \quad \|A\|_{1,\alpha} = |(1+|x|)^\alpha \nabla A|.$$

Since  $\dot{H}_s^1 \subset L^6(\mathbb{R}^3, \mathbb{R}^3)$ , we have  $\dot{H}_s^1 \subset H_{s,\alpha}^0$ ,  $\alpha < -1$ . Set

$$\mathcal{E}_\alpha = H_{s,\alpha}^0 \oplus \dot{H}_{s,\alpha}^1 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3, \quad \|Y\|_\alpha = \|E_s\|_{0,\alpha} + \|A\|_{1,\alpha} + |q| + |P|, \quad Y \in \mathcal{E}_\alpha.$$

For the fields we set

$$\mathcal{F}_\alpha = H_{s,\alpha}^0 \oplus \dot{H}_{s,\alpha}^1, \quad \|(E_s, A)\|_\alpha = \|E_s\|_{0,\alpha} + \|A\|_{1,\alpha}.$$

By (2.3) and the neutrality condition (1.14) we obtain that  $E_v \in H_{s,\alpha}^0$  for  $\alpha < 3/2$ ,  $A_v \in \dot{H}_{s,\alpha}^1$  for  $\alpha < 1/2$  and thus,

$$S_{a,v} \in \mathcal{E}_\alpha, \quad \text{for } \alpha < 1/2, \quad \text{where } S_{a,v} := (E_v(x-a), A_v(x-a), a, P_v). \quad (2.8)$$

The main result of our paper is the following theorem.

**Theorem 2.2** *Let the condition (1.12), Wiener condition (1.13), and the neutrality condition (1.14) hold, let  $Y^0 \in \mathcal{E}_\beta$  with  $\beta = 1 + \delta$ ,  $0 < \delta < 1/2$ . Suppose that  $Y^0$  is sufficiently close to the solitary manifold:*

$$Y^0 = S_{a_0,v_0} + Z_0, \quad d_\beta := \|Z_0\|_\beta \ll 1. \quad (2.9)$$

*Let  $Y(t) \in C(\mathbb{R}, \mathcal{E})$  be the solution to the Cauchy problem (2.1). Then the asymptotics hold for  $t \rightarrow \pm\infty$ ,*

$$\dot{q}(t) = v_\pm + \mathcal{O}(|t|^{-1-\delta}), \quad q(t) = v_\pm t + a_\pm + \mathcal{O}(|t|^{-2\delta}); \quad (2.10)$$

$$(E_s(x, t), A(x, t)) = (E_{v_\pm}(x - v_\pm t - a_\pm), A_{v_\pm}(x - v_\pm t - a_\pm)) + W^0(t) \Psi_\pm + r_\pm(x, t) \quad (2.11)$$

*with*

$$\|r_\pm(t)\|_{\mathcal{F}} = \mathcal{O}(|t|^{-\delta}). \quad (2.12)$$

It suffices to prove the asymptotics (2.11), (2.10) for  $t \rightarrow +\infty$  since the system (1.5)-(1.4) is time reversible.

### 3 Symplectic Projection

#### 3.1 Symplectic Structure

From now and later on we omit the subindex  $s$  in hamiltonian variables and write  $Y = (E, A, q, P)$ . The system (1.5) to (1.4) reads as the Hamiltonian system

$$\dot{Y} = J \mathcal{D}\mathcal{H}(Y), \quad J := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad Y = (E, A, q, P) \in \mathcal{E}, \quad (3.1)$$

where  $\mathcal{D}\mathcal{H}$  is the Fréchet derivative of the Hamilton functional (1.1). Let us identify the tangent space to  $\mathcal{E}$ , at every point, with  $\mathcal{E}$ . Consider the symplectic form  $\Omega$  defined on  $\mathcal{E}$  by

$$\Omega = \int dE(x) \wedge dA(x) dx + dq \wedge dP,$$

i.e.

$$\Omega(Y_1, Y_2) = \int (A_1 \cdot E_2 - A_2 \cdot E_1) dx + q_1 \cdot P_2 - q_2 \cdot P_1 \quad (3.2)$$

for  $Y_k = (E_k, A_k, q_k, P_k) \in \mathcal{E}$ ,  $k = 1, 2$  if the integral converges.

**Definition 3.1** *i)  $Y_1 \perp Y_2$  means that  $Y_1 \in \mathcal{E}$  is symplectic orthogonal to  $Y_2 \in \mathcal{E}$ , i.e.  $\Omega(Y_1, Y_2) = 0$ .*

*ii) A projection operator  $\mathbf{P} : \mathcal{E} \rightarrow \mathcal{E}$  is called symplectic orthogonal if  $Y_1 \perp Y_2$  for  $Y_1 \in \text{Ker } \mathbf{P}$  and  $Y_2 \in \text{Im } \mathbf{P}$ .*

### 3.2 Symplectic Projection onto Solitary Manifold

**Definition 3.2** A soliton state is  $S(\sigma) := (E_v(x - b), A_v(x - b), b, P_v)$ , where  $\sigma := (b, v)$  with  $b, v \in \mathbb{R}^3$  and  $|v| < 1$ .

Obviously, the soliton solution admits the representation  $S(\sigma(t))$ , where

$$\sigma(t) = (b(t), v(t)) = (vt + a, v). \quad (3.3)$$

**Definition 3.3** A solitary manifold is the set  $\mathcal{S} := \{S(b, v) : b \in \mathbb{R}^3, |v| < 1\}$ .

Let us consider the tangent space  $\mathcal{T}_{S(\sigma)}\mathcal{S}$  to the manifold  $\mathcal{S}$  at a point  $S(\sigma)$ . The vectors  $\tau_j := \partial_{\sigma_j} S(\sigma)$ , where  $\partial_{\sigma_j} := \partial_{b_j}$  and  $\partial_{\sigma_{j+3}} := \partial_{v_j}$  with  $j = 1, 2, 3$ , form a basis in  $\mathcal{T}_{\sigma}\mathcal{S}$ . In detail,

$$\begin{aligned} \tau_j = \tau_j(v) &:= \partial_{b_j} S(\sigma) = (-\partial_j E_v(y), -\partial_j A_v(y), e_j, 0) \\ \tau_{j+3} = \tau_{j+3}(v) &:= \partial_{v_j} S(\sigma) = (\partial_{v_j} E_v(y), \partial_{v_j} A_v(y), 0, \partial_{v_j} P_v) \end{aligned} \quad \left| \quad j = 1, 2, 3, \quad (3.4) \right.$$

where  $y := x - b$  is the “moving frame coordinate”,  $e_1 = (1, 0, 0)$  etc. Let us stress that the functions  $\tau_j$  will be considered always as the functions of  $y$ , not of  $x$ .

Note that the symplectic form  $\Omega(\tau_l(v), \tau_j(v))$  is well defined by the neutrality condition (1.14).

**Lemma 3.4** The matrix with the elements  $\Omega(\tau_l(v), \tau_j(v))$  is non-degenerate for  $|v| < 1$ .

The proof is made by a straightforward computation, see Appendix A.

Now we show that in a small neighborhood of the soliton manifold  $\mathcal{S}$  a “symplectic orthogonal projection” onto  $\mathcal{S}$  is well-defined. Let us introduce the translations

$$T_a : (\psi(\cdot), \pi(\cdot), q, p) \mapsto (\psi(\cdot - a), \pi(\cdot - a), q + a, p), \quad a \in \mathbb{R}^3.$$

Note that the manifold  $\mathcal{S}$  is invariant with respect to the translations.

**Definition 3.5** Let us denote by  $v(Y) := P/\sqrt{1+P^2}$  where  $P \in \mathbb{R}^3$  is the last component of the vector  $Y$ .

**Lemma 3.6** Let (1.12) hold,  $-3/2 < \alpha < 1/2$  and  $\bar{v} < 1$ . Then

i) there exists a neighborhood  $\mathcal{O}_\alpha(\mathcal{S})$  of  $\mathcal{S}$  in  $\mathcal{E}_\alpha$  and a map  $\Pi : \mathcal{O}_\alpha(\mathcal{S}) \rightarrow \mathcal{S}$  such that  $\Pi$  is uniformly continuous on  $\mathcal{O}_\alpha(\mathcal{S}) \cap \{Y \in \mathcal{E}_\alpha : v(Y) \leq \bar{v}\}$  in the metric of  $\mathcal{E}_\alpha$ ,

$$\Pi Y = Y \quad \text{for } Y \in \mathcal{S}, \quad \text{and} \quad Y - S \notin \mathcal{T}_S \mathcal{S}, \quad \text{where } S = \Pi Y. \quad (3.5)$$

ii)  $\mathcal{O}_\alpha(\mathcal{S})$  is invariant with respect to the translations  $T_a$ , and

$$\Pi T_a Y = T_a \Pi Y, \quad \text{for } Y \in \mathcal{O}_\alpha(\mathcal{S}) \quad \text{and} \quad a \in \mathbb{R}^3. \quad (3.6)$$

iii) For any  $\bar{v} < 1$  there exists a  $\tilde{v} < 1$  s.t.  $|v(\Pi Y)| < \tilde{v}$  when  $|v(Y)| < \bar{v}$ .

iv) For any  $\tilde{v} < 1$  there exists an  $r_\alpha(\tilde{v}) > 0$  s.t.  $S(\sigma) + Z \in \mathcal{O}_\alpha(\mathcal{S})$  if  $|v(S(\sigma))| < \tilde{v}$  and  $\|Z\|_\alpha < r_\alpha(\tilde{v})$ .

The proof is similar to that of Lemma 3.4 in [20].

We will call  $\Pi$  the symplectic orthogonal projection onto  $\mathcal{S}$ .

**Corollary 3.7** The condition (2.9) implies that  $Y_0 = S + Z_0$  where  $S = S(\sigma_0) = \Pi Y_0$ , and

$$\|Z_0\|_\beta \ll 1. \quad (3.7)$$



## 4 Linearization on the Solitary Manifold

Let us consider a solution to the system (1.5)–(1.4), and split it as the sum

$$Y(t) = S(\sigma(t)) + Z(t), \quad (4.1)$$

where  $\sigma(t) = (b(t), v(t)) \in \mathbb{R}^3 \times \{|v| < 1\}$  is an arbitrary smooth function of  $t \in \mathbb{R}$ . In detail, denote  $Y = (E, A, q, P)$  and  $Z = (e, a, r, \pi)$ . Then (4.1) means that

$$\begin{aligned} E(x, t) &= E_{v(t)}(x - b(t)) + e(x - b(t), t), & q(t) &= b(t) + r(t) \\ A(x, t) &= A_{v(t)}(x - b(t)) + a(x - b(t), t), & P(t) &= P_{v(t)} + \pi(t) \end{aligned} \quad (4.2)$$

Let us substitute (4.2) to (1.5)–(1.4) and linearize the equations in  $Z$ . Later we will choose  $S(\sigma(t)) = \Pi Y(t)$ , i.e.  $Z(t)$  is symplectic orthogonal to  $\mathcal{T}_{S(\sigma(t))}\mathcal{S}$ . However, this orthogonality condition is not needed for the formal process of linearization. The orthogonality condition will be important in Section 6, where we derive “modulation equations” for the parameters  $\sigma(t)$ .

Let us proceed to linearization. Setting  $y = x - b(t)$  which is the “moving frame coordinate”, we obtain from (4.2) and (1.2)–(1.4) that

$$\begin{aligned} \dot{E} &= \dot{v} \cdot \nabla_v E_{v(t)}(y) - \dot{b} \cdot \nabla E_{v(t)}(y) + \dot{e}(y, t) - \dot{b} \cdot \nabla e(y, t) \\ &= -\Delta(A_v(y) + a(y, t)) - \Pi_s(\rho(y - r)\dot{q}), \end{aligned} \quad (4.3)$$

$$\dot{A} = \dot{v} \cdot \nabla_v A_{v(t)}(y) - \dot{b} \cdot \nabla A_{v(t)}(y) + \dot{a}(y, t) - \dot{b} \cdot \nabla a(y, t) = -E_{v(t)}(y) - e(y, t), \quad (4.4)$$

$$\dot{q} = \dot{b} + \dot{r} = \frac{P_{v(t)} + \pi - \langle \rho(y - r), A_{v(t)}(y) + a(y, t) \rangle}{(1 + (P_{v(t)} + \pi - \langle \rho(y - r), A_{v(t)}(y) + a(y, t) \rangle)^2)^{1/2}}, \quad (4.5)$$

$$\dot{P} = \dot{v} \cdot \nabla_v P_{v(t)} + \dot{\pi} = \sum_k \dot{q}_k \langle \rho(y - r), \nabla(A_{v(t)}(y) + a(y, t))_k \rangle. \quad (4.6)$$

*Step i)* First we linearize the equation (4.5). Note that  $\rho(y - r) = \rho(y) - r \cdot \nabla \rho(y) + N_2(r)$ , where

$$|N_2(r)|_\beta \leq C_\beta(\bar{r})r^2, \quad (4.7)$$

uniformly in  $|r| \leq \bar{r}$  for any fixed  $\bar{r}$ . Here  $\beta$  is the constant defined in Theorem 2.2. Then (let us write  $v$  instead of  $v(t)$  and omit the other arguments for simplicity)

$$\langle \rho(y - r), A_v + a \rangle = \langle \rho, A_v \rangle + \langle \rho, a \rangle - \langle r \cdot \nabla \rho, A_v \rangle + N'_2 = \langle \rho, A_v \rangle + \langle \rho, a \rangle + N'_2, \quad (4.8)$$

where  $N'_2(r, a) = -\langle r \cdot \nabla \rho, a \rangle + \langle N_2, A_v + a \rangle$ . Here we use the equality  $\langle r \cdot \nabla \rho, A_v \rangle = 0$  which holds, since  $A_v$  is even and  $\nabla \rho$  is odd. Further, since  $P_v - \langle \rho, A_v \rangle = p_v$  by (2.3), we get  $P_v + \pi - \langle \rho(y - r), A_v + a \rangle = P_v + \pi - \langle \rho, A_v \rangle - \langle \rho, a \rangle - N'_2 = p_v + \pi - \langle \rho, a \rangle - N'_2 = p_v + s$ , where  $s := \pi - \langle \rho, a \rangle - N'_2$ . Applying Taylor expansion we obtain

$$(1 + (p_v + s)^2)^{-1/2} = \frac{1}{(1 + p_v^2)^{1/2}} - \frac{p_v \cdot s}{(1 + p_v^2)^{3/2}} + N'_3 = \frac{1}{(1 + p_v^2)^{1/2}} - \frac{v \cdot s}{1 + p_v^2} + N'_3,$$

since  $p_v/(1 + p_v^2)^{1/2} = v$ . Finally,

$$\begin{aligned} &\frac{P_v + \pi - \langle \rho(y - r), A_v + a \rangle}{(1 + (P_v + \pi - \langle \rho(y - r), A_v(y) + a(y, t) \rangle)^2)^{1/2}} = \frac{p_v + s}{(1 + (p_v + s)^2)^{1/2}} \\ &= v - \frac{(v, s)v}{(1 + (p_v)^2)^{1/2}} + \frac{s}{(1 + (p_v)^2)^{1/2}} + N''_3 = v + \nu(s - (v \cdot s)v) + N''_3, \end{aligned}$$

where  $\nu = \nu_v = (1 - v^2)^{1/2} = (1 + p_v^2)^{-1/2}$ . Insert the expression for  $s$ , then the equation (4.5) becomes

$$\dot{r} = v - \dot{b} + B_v(\pi - \langle \rho, a \rangle) + N_3, \quad (4.9)$$

where  $B_v := \nu(E - v \otimes v)$ , and

$$|N_3(Z)| \leq C(\tilde{v})|Z|_{-\beta}^2 \quad (4.10)$$

uniformly in  $|v| \leq \tilde{v} < 1$ .

*Step ii)* Next we linearize the equation (4.3). By (4.8) and (4.9) we obtain  $\rho(y - r)\dot{q} = \rho v + \rho B_v(\pi - \langle \rho, a \rangle) - r \cdot \nabla \rho v + N'_1$ . Substitute to the equation (4.3) and take (2.4) into account, then we get

$$\dot{e} = \dot{b} \cdot \nabla e - \Delta a + (\dot{b} - v) \cdot \nabla E_v - \dot{v} \cdot \nabla_v E_v - \Pi_s(\rho B_v(\pi - \langle \rho, a \rangle) - r \cdot \nabla \rho v) + N_1, \quad (4.11)$$

where for  $N_1$  the same bound (4.10) holds.

*Step iii)* Further, by (2.5) the equation (4.4) becomes

$$\dot{a} = -e + \dot{b} \cdot \nabla a + (\dot{b} - v) \cdot \nabla A_v - \dot{v} \cdot \nabla_v A_v. \quad (4.12)$$

*Step iv)* Let us proceed to the equation (4.6). We have

$$\begin{aligned} \dot{P} &= \dot{v} \cdot \nabla_v P_v + \dot{\pi} = \langle \rho(y - r), \dot{q} \cdot \nabla(A_v + a) \rangle \\ &= \langle \rho - r \cdot \nabla \rho + N_2, (v + B_v(\pi - \langle \rho, a \rangle) + N_3) \cdot \nabla(A_v + a) \rangle \\ &= \langle \rho - r \cdot \nabla \rho + N_2, v \cdot \nabla(A_v + a) + B_v(\pi - \langle \rho, a \rangle) \cdot \nabla A_v + N'_4 \rangle \\ &= \langle \rho, v \cdot \nabla(A_v + a) \rangle + \langle \rho, B_v(\pi - \langle \rho, a \rangle) \cdot \nabla A_v \rangle - \langle r \cdot \nabla \rho, v \cdot \nabla A_v \rangle + N_4 \\ &= \langle \rho, v \cdot \nabla a \rangle - \langle r \cdot \nabla \rho, v \cdot \nabla A_v \rangle + N_4, \end{aligned}$$

since  $\langle \rho, v \cdot \nabla A_v \rangle = 0$  and  $\langle \rho, B_v(\pi - \langle \rho, a \rangle) \cdot \nabla A_v \rangle = 0$ . Finally, the equation becomes

$$\dot{\pi} = \langle \rho, v \cdot \nabla a \rangle - \dot{v} \cdot \nabla_v P_v - \langle r \cdot \nabla \rho, v \cdot \nabla A_v \rangle + N_4, \quad (4.13)$$

where for  $N_4(v, Z)$  the estimate similar to (4.10) holds.

We can write the equations (4.9), (4.11)–(4.13) as

$$\dot{Z}(t) = A(t)Z(t) + T(t) + N(t), \quad t \in \mathbb{R}. \quad (4.14)$$

Here the operator  $A(t)$  depends on  $\sigma(t) = (b(t), v(t))$ . We will use the parameters  $v = v(t)$  and  $w := \dot{b}(t)$ . Then  $A(t)$  can be written in the form

$$\begin{aligned} A(t) \begin{pmatrix} e \\ a \\ r \\ \pi \end{pmatrix} &= A_{v,w} \begin{pmatrix} e \\ a \\ r \\ \pi \end{pmatrix} := \\ &\begin{pmatrix} w \cdot \nabla & -\Delta + \Pi_s(\rho B_v \langle \rho, \cdot \rangle) & \Pi_s(\cdot \nabla \rho v) & -\Pi_s(\rho B_v \cdot) \\ -1 & w \cdot \nabla & 0 & 0 \\ 0 & -B_v \langle \rho, \cdot \rangle & 0 & B_v \\ 0 & \langle \rho, v \cdot \nabla \cdot \rangle & -\langle \cdot \nabla \rho, v \cdot \nabla A_v \rangle & 0 \end{pmatrix} \begin{pmatrix} e \\ a \\ r \\ \pi \end{pmatrix}. \end{aligned} \quad (4.15)$$

Furthermore,  $T(t)$  and  $N(t)$  in (4.14) stand for

$$T(t) = T_{v,w} = \begin{pmatrix} (w - v) \cdot \nabla E_v - \dot{v} \cdot \nabla_v E_v \\ (w - v) \cdot \nabla A_v - \dot{v} \cdot \nabla_v A_v \\ v - w \\ -\dot{v} \cdot \nabla_v P_v \end{pmatrix}, \quad N(t) = N(\sigma, Z) = \begin{pmatrix} N_1(v, Z) \\ N_2(Z) \\ N_3(v, Z) \\ N_4(v, Z) \end{pmatrix}, \quad (4.16)$$

where  $v = v(t)$ ,  $w = w(t)$ ,  $\sigma = \sigma(t) = (b(t), v(t))$ , and  $Z = Z(t)$ . The estimates (4.7) and (4.10), which holds as well for  $N_1$  and  $N_4$  imply that

$$\|N(\sigma, Z)\|_\beta \leq C(\tilde{v}, \bar{Q})\|Z\|_{-\beta}^2 \quad (4.17)$$

uniformly in  $\sigma$  and  $\|Z\|_{-\beta} \leq r_{-\beta}(\tilde{v})$  for any fixed  $\tilde{v} < 1$  and  $|v| < \tilde{v}$ .

**Remarks 4.1** i) The term  $A(t)Z(t)$  in the right hand side of the equation (4.14) is linear in  $Z(t)$ , and  $N(t)$  is a *high order term* in  $Z(t)$ . The term  $T(t)$  vanishes if  $S(\sigma(t))$  is a soliton solution since in this case  $\dot{v} = 0$  and  $w = \dot{b} = v$ . Otherwise  $T(t)$  is a zero order term which does not vanish though  $S(\sigma(t))$  belongs to the solitary manifold.

ii) Formulas (3.4) and (4.16) imply:

$$T(t) = - \sum_{l=1}^3 [(w - v)_l \tau_l + \dot{v}_l \tau_{l+3}] \quad (4.18)$$

and hence  $T(t) \in \mathcal{T}_{S(\sigma(t))}\mathcal{S}$ ,  $t \in \mathbb{R}$ .

## 5 The Linearized Equation

Here we study some properties of the operator (4.15). First, let us compute the action of  $A_{v,w}$  on the tangent vectors  $\tau_j$  to the solitary manifold  $\mathcal{S}$ .

**Lemma 5.1** *The operator  $A_{v,w}$  acts on the tangent vectors  $\tau_j(v)$  to the solitary manifold as follows,*

$$A_{v,w}[\tau_j(v)] = (w - v) \cdot \nabla \tau_j(v), \quad A_{v,w}[\tau_{j+3}(v)] = (w - v) \cdot \nabla \tau_{j+3}(v) + \tau_j(v), \quad j = 1, 2, 3. \quad (5.1)$$

**Proof** We differentiate the stationary equations (2.4) to (2.7) and obtain

$$\partial_j E_v = v \cdot \nabla \partial_j A_v, \quad v \cdot \nabla \partial_j E_v = \Delta \partial_j A_v + \Pi_s(\partial_j \rho v), \quad (5.2)$$

$$\partial_{v_j} E_v = \partial_j A_v + v \cdot \nabla \partial_{v_j} A_v, \quad \partial_j E_v + v \cdot \nabla \partial_{v_j} E_v = \Delta \partial_{v_j} A_v + \Pi_s(\rho e_j), \quad (5.3)$$

$$e_j = B_v \partial_{v_j} p_v = B_v \partial_{v_j} (P_v - \langle \rho, A_v \rangle), \quad 0 = \langle \rho, e_j \cdot \nabla A_v \rangle + \langle \rho, v \cdot \nabla \partial_{v_j} A_v \rangle. \quad (5.4)$$

Let us apply  $A_{v,w}$  to  $\tau_j$ ,  $j = 1, 2, 3$ , i.e. to  $(-\partial_j E_v, -\partial_j A_v, e_j, 0)$ . The first component equals

$$w \cdot \nabla (-\partial_j E_v) - \Delta (-\partial_j A_v) + \Pi_s(\rho B_v \langle \rho, -\partial_j A_v \rangle) + \Pi_s(\partial_j \rho v) = (w - v) \cdot \nabla (-\partial_j E_v) \quad (5.5)$$

by (5.2) and since  $\langle \rho, \partial_j A_v \rangle = 0$ . The second component equals

$$\partial_j E_v - w \cdot \nabla (\partial_j A_v) = (w - v) \cdot \nabla (-\partial_j A_v) \quad (5.6)$$

by (5.2). The third component equals

$$-B_v \langle \rho, -\partial_j A_v \rangle = 0, \quad (5.7)$$

and the last is

$$\langle \rho, v \cdot \nabla (-\partial_j A_v) \rangle - \langle \rho, B_v \langle \rho, -\partial_j A_v \rangle \cdot \nabla A_v \rangle - \langle \partial_j \rho, v \cdot \nabla A_v \rangle = 0. \quad (5.8)$$

The formulas (5.5) to (5.8) mean  $A_{v,w}[\tau_j] = (w - v) \cdot \nabla \tau_j$ ,  $j = 1, 2, 3$ .

Now let us apply  $A_{v,w}$  to  $\tau_{j+3}$ ,  $j = 1, 2, 3$ , i.e. to  $(\partial_{v_j} E_v, \partial_{v_j} A_v, 0, \partial_{v_j} P_v)$ . The first component equals

$$\begin{aligned} & w \cdot \nabla \partial_{v_j} E_v - \Delta (\partial_{v_j} A_v) + \Pi_s(\rho B_v \langle \rho, \partial_{v_j} A_v \rangle) \\ &= (w - v) \cdot \nabla \partial_{v_j} E_v - \partial_j E_v + \Pi_s(\rho e_j) + \Pi_s(\rho B_v \langle \rho, \partial_{v_j} A_v \rangle) - \Pi_s(\rho B_v \partial_{v_j} P_v) \end{aligned}$$

$$= (w - v) \cdot \nabla \partial_{v_j} E_v - \partial_j E_v \quad (5.9)$$

by (5.3) and (5.4). The second component equals

$$- \partial_{v_j} E_v + w \cdot \nabla \partial_{v_j} A_v = (w - v) \cdot \nabla \partial_{v_j} A_v - \partial_j A_v \quad (5.10)$$

by (5.3). The third component equals

$$- B_v \langle \rho, \partial_{v_j} A_v \rangle + B_v \partial_{v_j} P_v = B_v \partial_{v_j} p_v = e_j \quad (5.11)$$

by (5.4), and the last is

$$\begin{aligned} & \langle \rho, v \cdot \nabla \partial_{v_j} A_v \rangle - \langle \rho, B_v \langle \rho, \partial_{v_j} A_v \rangle \cdot \nabla A_v \rangle + \langle \rho, (B_v \partial_{v_j} P_v) \cdot \nabla A_v \rangle \\ &= \langle \rho, v \cdot \nabla \partial_{v_j} A_v \rangle - \langle \rho, B_v (\partial_{v_j} P_v - \partial_{v_j} p_v) \cdot \nabla A_v \rangle + \langle \rho, (B_v \partial_{v_j} P_v) \cdot \nabla A_v \rangle \\ &= \langle \rho, v \cdot \nabla \partial_{v_j} A_v \rangle + \langle \rho, \partial_j A_v \rangle = 0 \end{aligned} \quad (5.12)$$

by (5.4). The formulas (5.9) to (5.12) mean  $A_{v,w}[\tau_{j+3}] = (w - v) \cdot \nabla \tau_{j+3} + \tau_j$ ,  $j = 1, 2, 3$ .  $\square$

Consider the linear equation

$$\dot{X}(t) = A_{v,w} X(t), \quad t \in \mathbb{R} \quad (5.13)$$

with an arbitrary fixed  $v$  such that  $|v| < 1$  and  $w \in \mathbb{R}^3$ .

**Lemma 5.2** *i) For any  $v$ ,  $|v| < 1$ ,  $w \in \mathbb{R}^3$  the equation (5.13) formally can be written as the Hamiltonian system (cf. (3.1)),*

$$\dot{X}(t) = J D\mathcal{H}_{v,w}(X(t)), \quad t \in \mathbb{R}, \quad (5.14)$$

where  $D\mathcal{H}_v$  is the Fréchet derivative of the Hamilton functional

$$\begin{aligned} \mathcal{H}_{v,w}(X) &= \frac{1}{2} \int \left[ |e|^2 + |\nabla a|^2 \right] dy + \int a(w \cdot \nabla) e dy + \frac{1}{2} (B_v \langle \rho, a \rangle) \cdot \langle \rho, a \rangle \\ &+ \frac{1}{2} \pi \cdot B_v \pi + \langle r \cdot \nabla \rho v, a \rangle - \langle \rho B_v \pi, a \rangle + \frac{1}{2} \langle r \cdot \nabla \rho, v \cdot (\nabla A_v) \rangle, \quad X = (e, a, r, \pi) \in \mathcal{E}. \end{aligned} \quad (5.15)$$

*ii) Energy conservation law holds for the solutions  $X(t) \in C^1(\mathbb{R}, \mathcal{E})$ ,*

$$\mathcal{H}_{v,w}(X(t)) = \text{const}, \quad t \in \mathbb{R}. \quad (5.16)$$

*iii) The skew-symmetry relation holds,*

$$\Omega(A_{v,w} X_1, X_2) = -\Omega(X_1, A_{v,w} X_2), \quad X_1, X_2 \in \mathcal{E}. \quad (5.17)$$

We will apply Lemma 5.2 mainly to the operator  $A_{v,v}$  corresponding to  $w = v$ . In that case the linearized equation has the following additional essential features.

**Lemma 5.3** *Let us assume that  $w = v$  and  $|v| < 1$ . Then*

*i) The tangent vectors  $\tau_j(v)$  with  $j = 1, 2, 3$  are eigenvectors, and  $\tau_{j+3}(v)$  are root vectors of the operator  $A_{v,v}$ , corresponding to zero eigenvalue, i.e.*

$$A_{v,v}[\tau_j(v)] = 0, \quad A_{v,v}[\tau_{j+3}(v)] = \tau_j(v), \quad j = 1, 2, 3. \quad (5.18)$$

*ii) The Hamilton function (5.15) is positive definite,*

$$\mathcal{H}_{v,v}(X) \geq 0. \quad (5.19)$$

**Proof** The first statement follows from (5.1). To prove the second statement let us rewrite

$$\begin{aligned}
\mathcal{H}_{v,v}(X) &= \frac{1}{2} \int \left[ |e|^2 + |\nabla a|^2 \right] dy + \int a(v \cdot \nabla) e dy + \frac{1}{2} (B_v \langle \rho, a \rangle) \cdot \langle \rho, a \rangle \\
&+ \frac{1}{2} \pi \cdot B_v \pi + \langle r \cdot \nabla \rho v, a \rangle - \langle \rho B_v \pi, a \rangle + \frac{1}{2} \langle r \cdot \nabla \rho, v \cdot (\nabla A_v) \rangle, \quad X = (e, a, r, \pi) \in \mathcal{E}. \\
&= \frac{1}{2} (B_v (\pi - \langle \rho, a \rangle)) \cdot (\pi - \langle \rho, a \rangle) + \frac{1}{2} (\langle e, e \rangle + \langle (v \cdot \nabla) a, (v \cdot \nabla) a \rangle - \langle e, (v \cdot \nabla) a \rangle) \\
&+ \frac{1}{2} (\langle (-\Delta + (v \cdot \nabla)^2) a, a \rangle + \langle (r \cdot \nabla) \rho v, a \rangle + \langle (r \cdot \nabla) \rho, v \cdot (\nabla A_v) \rangle).
\end{aligned}$$

Here the first line is clearly nonnegative, since  $B_v$  is nonnegative definite. The last line in Fourier space by (A.4) equals

$$\frac{1}{2} \int \left( (k^2 - (kv)^2) |\hat{a}|^2 - 2i(kr) \hat{\rho}(v \cdot \bar{\hat{a}}) + \frac{(kr)^2 |\hat{\rho}|^2 v^2}{k^2 - (kv)^2} \right) dk.$$

The integrand is nonnegative, since

$$\int \operatorname{Im} [2i(kr) \hat{\rho}(v \cdot \bar{\hat{a}})] dk \in \mathbb{R} \quad \text{and} \quad |\operatorname{Re} [i(kr) \hat{\rho}(v \cdot \bar{\hat{a}})]| \leq |(kr)| |\hat{\rho}| |v| |\hat{a}|.$$

This completes the proof.  $\square$

**Remark 5.4** Lemma 5.3 ii) together with energy conservation (5.16) will imply the analyticity of the resolvent  $(A_{v,v} - \lambda)^{-1}$  for  $\operatorname{Re} \lambda > 0$ , see Section 14.

**Remark 5.5** For a soliton solution of the system (1.5)–(1.4) we have  $\dot{b} = v$ ,  $\dot{v} = 0$ , and hence  $T(t) \equiv 0$ . Thus, the equation (5.13) is the linearization of the system (1.5)–(1.4) on a soliton solution. In fact, we do not linearize (1.5)–(1.4) on a soliton solution, but on a trajectory  $S(\sigma(t))$  with  $\sigma(t)$  being nonlinear in  $t$ . We will show later that  $T(t)$  is quadratic in  $Z(t)$  if we choose  $S(\sigma(t))$  to be the symplectic orthogonal projection of  $Y(t)$ . Then (5.13) is again the linearization of (1.5)–(1.4).

## 6 Symplectic Decomposition of the Dynamics

Here we decompose the dynamics in two components: along the manifold  $\mathcal{S}$  and in transversal directions. The equation (4.14) is obtained without any assumption on  $\sigma(t)$  in (4.1). We are going to choose  $S(\sigma(t)) := \Pi Y(t)$ , but then we need to know that

$$Y(t) \in \mathcal{O}_\alpha(\mathcal{S}), \quad t \in \mathbb{R}, \tag{6.1}$$

with some  $\mathcal{O}_\alpha(\mathcal{S})$  defined in Lemma 3.5. It is true for  $t = 0$  by our main assumption (2.9) with sufficiently small  $d_\beta > 0$ . Then  $S(\sigma(0)) = \Pi Y(0)$  and  $Z(0) = Y(0) - S(\sigma(0))$  are well defined. We will prove below that (6.1) holds with  $\alpha = -\beta$  if  $d_\beta$  is sufficiently small. First, the a priori estimate (2.2) together with Lemma 3.6 iii) imply that  $\Pi Y(t) = S(\sigma(t))$  with  $\sigma(t) = (b(t), v(t))$ , and

$$|v(t)| \leq \tilde{v} < 1, \quad t \in \mathbb{R} \tag{6.2}$$

if  $Y(t) \in \mathcal{O}_{-\beta}(\mathcal{S})$ . Denote by  $r_{-\beta}(\tilde{v})$  the positive number from Lemma 3.6 iv) which corresponds to  $\alpha = -\beta$ . Then  $S(\sigma) + Z \in \mathcal{O}_{-\beta}(\mathcal{S})$  if  $\sigma = (b, v)$  with  $|v| < \tilde{v}$  and  $\|Z\|_{-\beta} < r_{-\beta}(\tilde{v})$ . Note that (2.2) implies  $\|Z(0)\|_{-\beta} < r_{-\beta}(\tilde{v})$  if  $d_\beta$  is sufficiently small. Therefore,  $S(\sigma(t)) = \Pi Y(t)$  and  $Z(t) = Y(t) - S(\sigma(t))$  are well defined for  $t \geq 0$  so small that  $\|Z(t)\|_{-\beta} < r_{-\beta}(\tilde{v})$ . This is formalized by the following standard definition.

**Definition 6.1**  $t_*$  is the “exit time”,

$$t_* = \sup\{t > 0 : \|Z(s)\|_{-\beta} < r_{-\beta}(\tilde{v}), \quad 0 \leq s \leq t\}, \quad Z(s) = Y(s) - S(\sigma(s)). \quad (6.3)$$

One of our main goals is to prove that  $t_* = \infty$  if  $d_\beta$  is sufficiently small. This would follow if we show that

$$\|Z(t)\|_{-\beta} < r_{-\beta}(\tilde{v})/2, \quad 0 \leq t < t_*. \quad (6.4)$$

Note that

$$|r(t)| \leq \bar{r} := r_{-\beta}(\tilde{v}), \quad 0 \leq t < t_*. \quad (6.5)$$

Now  $N(t)$  in (4.14) satisfies, by (4.17), the following estimate,

$$\|N(t)\|_\beta \leq C_\beta(\tilde{v})\|Z(t)\|_{-\beta}^2, \quad 0 \leq t < t_*. \quad (6.6)$$

## 6.1 Longitudinal Dynamics: Modulation Equations

From now on we fix the decomposition  $Y(t) = S(\sigma(t)) + Z(t)$  for  $0 < t < t_*$  by setting  $S(\sigma(t)) = \Pi Y(t)$  which is equivalent to the symplectic orthogonality condition of type (3.5),

$$Z(t) \dagger \mathcal{T}_{S(\sigma(t))}\mathcal{S}, \quad 0 \leq t < t_*. \quad (6.7)$$

This allows us to simplify drastically the asymptotic analysis of the dynamical equations (4.14) for the transversal component  $Z(t)$ . As the first step, we derive the longitudinal dynamics, i.e. the “modulation equations” for the parameters  $\sigma(t)$ . Let us derive a system of ordinary differential equations for the vector  $\sigma(t)$ . For this purpose, let us write (6.7) in the form

$$\Omega(Z(t), \tau_j(t)) = 0, \quad j = 1, \dots, 6, \quad 0 \leq t < t_*, \quad (6.8)$$

where the vectors  $\tau_j(t) = \tau_j(\sigma(t))$  span the tangent space  $\mathcal{T}_{S(\sigma(t))}\mathcal{S}$ . Note that  $\sigma(t) = (b(t), v(t))$ , where

$$|v(t)| \leq \tilde{v} < 1, \quad 0 \leq t < t_*, \quad (6.9)$$

by Lemma 3.6 iii). It would be convenient for us to use some other parameters  $(c, v)$  instead of  $\sigma = (b, v)$ , where  $c(t) = b(t) - \int_0^t v(\tau) d\tau$  and

$$\dot{c}(t) = \dot{b}(t) - v(t) = w(t) - v(t), \quad 0 \leq t < t_*. \quad (6.10)$$

We do not need an explicit form of the equations for  $(c, v)$  but the following statement.

**Lemma 6.2** (cf. [20], Lemma 6.2) *Let  $Y(t)$  be a solution to the Cauchy problem (2.1), and (4.1), (6.8) hold. Then  $(c(t), v(t))$  satisfies the equation*

$$\begin{pmatrix} \dot{c}(t) \\ \dot{v}(t) \end{pmatrix} = \mathcal{N}(\sigma(t), Z(t)), \quad 0 \leq t < t_*, \quad (6.11)$$

where

$$\mathcal{N}(\sigma, Z) = \mathcal{O}(\|Z\|_{-\beta}^2) \quad (6.12)$$

uniformly in  $\sigma \in \Sigma(\tilde{v}) := \{(b, v) : |v| \leq \tilde{v}\}$ .

**Proof** We differentiate the orthogonality conditions (6.8) in  $t$ , and obtain

$$0 = \Omega(\dot{Z}, \tau_j) + \Omega(Z, \dot{\tau}_j) = \Omega(AZ + T + N, \tau_j) + \Omega(Z, \dot{\tau}_j), \quad 0 \leq t < t_*. \quad (6.13)$$

First, let us compute the principal (i.e. non-vanishing at  $Z = 0$ ) term  $\Omega(T, \tau_j)$ . For  $j = 1, 2, 3$  one has by (4.18), (A.14)

$$\Omega(T, \tau_j) = - \sum_l (\dot{c}_l \Omega(\tau_l, \tau_j) + \dot{v}_l \Omega(\tau_{l+3}, \tau_j)) = \sum_l \Omega(\tau_j, \tau_{l+3}) \dot{v}_l = \sum_l \Omega_{jl}^+ \dot{v}_l,$$

where the matrix  $\Omega^+$  is defined by (A.13). Similarly,

$$\Omega(T, \tau_{j+3}) = - \sum_l (\dot{c}_l \Omega(\tau_l, \tau_{j+3}) + \dot{v}_l \Omega(\tau_{l+3}, \tau_{j+3})) = \sum_l \Omega(\tau_{j+3}, \tau_l) \dot{c}_l = - \sum_l \Omega_{jl}^+ \dot{c}_l.$$

As the result, we have by (A.14),

$$\Omega(T, \tau) = \begin{pmatrix} 0 & \Omega^+(v) \\ -\Omega^+(v) & 0 \end{pmatrix} \begin{pmatrix} \dot{c} \\ \dot{v} \end{pmatrix} = \Omega(v) \begin{pmatrix} \dot{c} \\ \dot{v} \end{pmatrix} \quad (6.14)$$

in the vector form.

Second, let us compute  $\Omega(AZ, \tau_j)$ . The skew-symmetry (5.17) implies that  $\Omega(AZ, \tau_j) = -\Omega(Z, A\tau_j)$ . Then for  $j = 1, 2, 3$ , we have by (5.1),

$$\Omega(AZ, \tau_j) = -\Omega(Z, \dot{c} \cdot \nabla \tau_j), \quad (6.15)$$

and similarly,

$$\Omega(AZ, \tau_{j+3}) = -\Omega(Z, \dot{c} \cdot \nabla \tau_{j+3} + \tau_j) = -\Omega(Z, \dot{c} \cdot \nabla \tau_{j+3}) - \Omega(Z, \tau_j) = -\Omega(Z, \dot{c} \cdot \nabla \tau_{j+3}), \quad (6.16)$$

since  $\Omega(Z, \tau_j) = 0$ .

Finally, let us compute the last term  $\Omega(Z, \dot{\tau}_j)$ . For  $j = 1, \dots, 6$  one has  $\dot{\tau}_j = \dot{b} \cdot \nabla_b \tau_j + \dot{v} \cdot \nabla_v \tau_j = \dot{v} \cdot \nabla_v \tau_j$  since the vectors  $\tau_j$  do not depend on  $b$  according to (3.4). Hence,

$$\Omega(Z, \dot{\tau}_j) = \Omega(Z, \dot{v} \cdot \nabla_v \tau_j). \quad (6.17)$$

As the result, by (6.14)-(6.17), the equation (6.13) becomes

$$0 = \Omega(v) \begin{pmatrix} \dot{c} \\ \dot{v} \end{pmatrix} + \mathcal{M}_0(\sigma, Z) \begin{pmatrix} \dot{c} \\ \dot{v} \end{pmatrix} + \mathcal{N}_0(\sigma, Z), \quad (6.18)$$

where the matrix  $\mathcal{M}_0(\sigma, Z) = \mathcal{O}(\|Z\|_{-\beta})$ , and  $\mathcal{N}_0(\sigma, Z) = \mathcal{O}(\|Z\|_{-\beta}^2)$  uniformly in  $\sigma \in \Sigma(\tilde{v})$  and  $\|Z\|_{-\beta} < r_{-\beta}(\tilde{v})$ . Then, since  $\Omega(v)$  is invertible by Lemma 3.4, and  $\|Z\|_{-\beta}$  is small, we can resolve (6.18) with respect to the derivatives and obtain equations (6.11) with  $\mathcal{N} = \mathcal{O}(\|Z\|_{-\beta}^2)$  uniformly in  $\sigma \in \Sigma(\tilde{v})$ .  $\square$

**Remark 6.3** The equations (6.11), (6.12) imply that the soliton parameters  $c(t)$  and  $v(t)$  are *adiabatic invariants* (see [2]).

## 6.2 Decay for the Transversal Dynamics

In Section 11 we will show that our main Theorem 2.2 can be derived from the following time decay of the transversal component  $Z(t)$ :

**Proposition 6.4** *Let all conditions of Theorem 2.2 hold. Then  $t_* = \infty$ , and*

$$\|Z(t)\|_{-\beta} \leq \frac{C(\rho, \bar{v}, d_\beta)}{(1 + |t|)^\beta}, \quad t \geq 0. \quad (6.19)$$

We will derive (6.19) in Sections 8-11 from our equation (4.14) for the transversal component  $Z(t)$ . This equation can be specified using Lemma 6.2. Indeed, the lemma implies that

$$\|T(t)\|_\beta \leq C(\tilde{v})\|Z(t)\|_{-\beta}^2, \quad 0 \leq t < t_*, \quad (6.20)$$

by (4.16) since  $w - v = \dot{c}$ . Thus (4.14) becomes the equation

$$\dot{Z}(t) = A(t)Z(t) + \tilde{N}(t), \quad 0 \leq t < t_*, \quad (6.21)$$

where  $A(t) = A_{v(t), w(t)}$ , and  $\tilde{N}(t) := T(t) + N(t)$  satisfies the estimate

$$\|\tilde{N}(t)\|_\beta \leq C\|Z(t)\|_{-\beta}^2, \quad 0 \leq t < t_*. \quad (6.22)$$

In all remaining part of our paper we will analyze mainly the **basic equation** (6.21) to establish the decay (6.19). We are going to derive the decay using the bound (6.22) and the orthogonality condition (6.7).

Let us comment on two main difficulties in proving (6.19). The difficulties are common for the problems studied in [7, 10]. First, the linear part of the equation is non-autonomous, hence we cannot apply directly known methods of scattering theory. Similarly to the approach of [7, 10], we reduce the problem to the analysis of the *frozen* linear equation,

$$\dot{X}(t) = A_1 X(t), \quad t \in \mathbb{R}, \quad (6.23)$$

where  $A_1$  is the operator  $A_{v_1, v_1}$  defined in (4.15) with  $v_1 = v(t_1)$  and a fixed  $t_1 \in [0, t_*)$ . Then we estimate the error by the method of majorants.

Second, even for the frozen equation (6.23), the decay of type (6.19) for all solutions does not hold without the orthogonality condition of type (6.7). Namely, by (5.18) the equation (6.23) admits the *secular solutions*

$$X(t) = \sum_1^3 C_j \tau_j(v_1) + \sum_1^3 D_j [\tau_j(v_1)t + \tau_{j+3}(v_1)] \quad (6.24)$$

which arise also by differentiation of the soliton (1.8) in the parameters  $a$  and  $v_1$  in the moving coordinate  $y = x - v_1 t$ . Hence, we have to take into account the orthogonality condition (6.7) in order to avoid the secular solutions. For this purpose we will apply the corresponding symplectic orthogonal projection which kills the “runaway solutions” (6.24).

**Remark 6.5** The solution (6.24) lies in the tangent space  $\mathcal{T}_{S(\sigma_1)}\mathcal{S}$  with  $\sigma_1 = (b_1, v_1)$  (for an arbitrary  $b_1 \in \mathbb{R}$ ) that suggests an unstable character of the nonlinear dynamics *along the solitary manifold* (cf. Remark 4.1 iii)).

**Definition 6.6** i) Denote by  $\Pi_v$ ,  $|v| < 1$ , the symplectic orthogonal projection of  $\mathcal{E}$  onto the tangent space  $\mathcal{T}_{S(\sigma)}\mathcal{S}$ , and  $\mathbf{P}_v = \mathbf{I} - \Pi_v$ .

ii) Denote by  $\mathcal{Z}_v = \mathbf{P}_v \mathcal{E}$  the space symplectic orthogonal to  $\mathcal{T}_{S(\sigma)}\mathcal{S}$  with  $\sigma = (b, v)$  (for an arbitrary  $b \in \mathbb{R}$ ).

Note that by the linearity,

$$\Pi_v Z = \sum \Pi_{jl}(v) \tau_j(v) \Omega(\tau_l(v), Z), \quad Z \in \mathcal{E}, \quad (6.25)$$

with some smooth coefficients  $\Pi_{jl}(v)$ . Hence, the projector  $\Pi_v$ , in the variable  $y = x - b$ , does not depend on  $b$ , and this explains the choice of the subindex in  $\Pi_v$  and  $\mathbf{P}_v$ .

Now we have the symplectic orthogonal decomposition

$$\mathcal{E}_\beta = \mathcal{T}_{S(\sigma)}\mathcal{S} + \mathcal{Z}_v, \quad \sigma = (b, v), \quad (6.26)$$

and the symplectic orthogonality (6.7) can be written in the following equivalent forms,

$$\Pi_{v(t)} Z(t) = 0, \quad \mathbf{P}_{v(t)} Z(t) = Z(t), \quad 0 \leq t < t_*. \quad (6.27)$$



**Remark 6.7** The tangent space  $\mathcal{T}_{S(\sigma)}\mathcal{S}$  is invariant under the operator  $A_{v,v}$  by Lemma 5.3 i), hence the space  $\mathcal{Z}_v$  is also invariant by (5.17):  $A_{v,v}Z \in \mathcal{Z}_v$  for *sufficiently smooth*  $Z \in \mathcal{Z}_v$ .

In the Sections 12-18 we prove the following proposition which is one of the main ingredients for proving (6.19). Let us consider the Cauchy problem for the equation (6.23) with  $A_1 = A_{v_1, v_1}$  for a fixed  $v_1$ ,  $|v_1| < 1$ . Recall that the parameter  $\beta = 1 + \delta$ ,  $0 < \delta < 1/2$  is also fixed.

**Proposition 6.8** *Let the Wiener condition (1.13) and the neutrality condition (1.14) hold,  $|v_1| \leq \tilde{v} < 1$ , and  $X_0 \in \mathcal{E}$ . Then*

i) *The equation (6.23), with  $A_1 = A_{v_1, v_1}$ , admits the unique solution  $e^{A_1 t} X_0 := X(t) \in C(\mathbb{R}, \mathcal{E})$  with the initial condition  $X(0) = X_0$ .*

ii) *For  $X_0 \in \mathcal{Z}_{v_1} \cap \mathcal{E}_\beta$ , the solution  $X(t)$  has the following decay,*

$$\|e^{A_1 t} X_0\|_{-\beta} \leq \frac{C_\beta(\tilde{v})}{(1 + |t|)^\beta} \|X_0\|_\beta, \quad t \in \mathbb{R}. \quad (6.28)$$

**Remark 6.9** The decay is provided by two fundamental facts which we will establish below:

i) the null root space of the generator  $A_1$  coincides with the  $\mathcal{T}_{S(\sigma_1)}\mathcal{S}$  with  $\sigma_1 = (b_1, v_1)$  (for an arbitrary  $b_1 \in \mathbb{R}$ ), and

ii) the spectrum of  $A_1$  in the space  $\mathcal{Z}_{v_1}$  is absolutely continuous.

## 7 Frozen Form of Transversal Dynamics

Now let us fix an arbitrary  $t_1 \in [0, t_*)$ , and rewrite the equation (6.21) in a “frozen form”

$$\dot{Z}(t) = A_1 Z(t) + (A(t) - A_1)Z(t) + \tilde{N}(t), \quad 0 \leq t < t_*, \quad (7.1)$$

where  $A_1 = A_{v(t_1), v(t_1)}$  and  $A(t) - A_1 =$

$$\begin{pmatrix} [w-v_1] \cdot \nabla & \Pi_s(\rho(B_v - B_{v_1})\langle \rho, \cdot \rangle) & \Pi_s(\cdot \nabla \rho(v - v_1)) & -\Pi_s(\rho(B_v - B_{v_1})\langle \cdot \rangle) \\ 0 & [w-v_1] \cdot \nabla & 0 & 0 \\ 0 & -(B_v - B_{v_1})\langle \rho, \cdot \rangle & 0 & B_v - B_{v_1} \\ 0 & \langle \rho, (v - v_1) \nabla \cdot \rangle & -\langle \cdot \nabla \rho, (v \nabla A_v - v_1 \nabla A_{v_1}) \rangle & 0 \end{pmatrix}$$

where  $w = w(t)$ ,  $v = v(t)$ ,  $v_1 = v(t_1)$ . The next trick is important since it allows us to kill the “bad terms”  $[w(t) - v(t_1)] \cdot \nabla$  in the operator  $A(t) - A_1$ .

**Definition 7.1** *Let us change the variables  $(y, t) \mapsto (y_1, t) = (y + d_1(t), t)$  where*

$$d_1(t) := \int_{t_1}^t (w(s) - v(t_1)) ds, \quad 0 \leq t \leq t_1. \quad (7.2)$$

Next define

$$\begin{aligned} Z_1(t) &= (e_1(y_1, t), a_1(y_1, t), r(t), \pi(t)) := (e(y, t), a(y, t), r(t), \pi(t)) \\ &= (e(y_1 - d_1(t), t), a(y_1 - d_1(t), t), r(t), \pi(t)). \end{aligned} \quad (7.3)$$

Then we obtain the final form of the “frozen equation” for the transversal dynamics

$$\dot{Z}_1(t) = A_1 Z_1(t) + B_1(t) Z_1(t) + N_1(t), \quad 0 \leq t \leq t_1, \quad (7.4)$$

where  $N_1(t) = \tilde{N}(t)$  is expressed in terms of  $y = y_1 - d_1(t)$ , and  $B_1(t) =$

$$\begin{pmatrix} 0 & \Pi_s(\rho(B_{v(t)} - B_{v(t_1)})\langle \rho, \cdot \rangle) & \Pi_s(\cdot \nabla \rho(v(t) - v(t_1))) & -\Pi_s(\rho(B_{v(t)} - B_{v(t_1)})\langle \cdot \rangle) \\ 0 & 0 & 0 & 0 \\ 0 & -(B_{v(t)} - B_{v(t_1)})\langle \rho, \cdot \rangle & 0 & B_{v(t)} - B_{v(t_1)} \\ 0 & \langle \rho, (v(t) - v(t_1)) \cdot \nabla \cdot \rangle & -\langle \cdot \nabla \rho, (v(t) \nabla A_{v(t)} - v(t_1) \nabla A_{v(t_1)}) \rangle & 0 \end{pmatrix}.$$

At the end of this section, we will derive appropriate bounds for the “remainder terms”  $B_1(t)Z_1(t)$  and  $N_1(t)$  in (7.4). First, note that we have by Lemma 6.2,

$$|B_{v(t)} - B_{v(t_1)}| \leq \left| \int_{t_1}^t \dot{v}(s) \cdot \nabla_v B_{v(s)} ds \right| \leq C \int_t^{t_1} \|Z(s)\|_{-\beta}^2 ds. \quad (7.5)$$

The similar estimates hold also for all the rest terms of  $B_1(t)$ . Let us recall the following well-known inequality: for any  $\alpha \in \mathbb{R}$

$$(1 + |y + x|)^\alpha \leq (1 + |y|)^\alpha (1 + |x|)^{|\alpha|}, \quad x, y \in \mathbb{R}^3. \quad (7.6)$$

**Lemma 7.2** *For  $(e, a, r, \pi) \in \mathcal{E}_\alpha$  with any  $\alpha \in \mathbb{R}$  the following estimate holds:*

$$\|(e(y_1 - d_1), a(y_1 - d_1), r, \pi)\|_\alpha \leq \|(e, a, r, \pi)\|_\alpha (1 + |d_1|)^{|\alpha|}, \quad d_1 \in \mathbb{R}^3. \quad (7.7)$$

For proof see [20].

**Corollary 7.3** *The following bound holds*

$$\|N_1(t)\|_\beta \leq \|Z_1(t)\|_{-\beta}^2 (1 + |d_1(t)|)^{3\beta}, \quad 0 \leq t \leq t_1. \quad (7.8)$$

Indeed, applying the previous lemma twice, once for  $\beta$  and once for  $-\beta$ , we obtain from (6.22) that

$$\|N_1(t)\|_\beta \leq (1 + |d_1(t)|)^\beta \|\tilde{N}(t, Z(t))\|_\beta \leq (1 + |d_1(t)|)^\beta \|Z\|_{-\beta}^2 \leq (1 + |d_1(t)|)^{3\beta} \|Z_1(t)\|_{-\beta}^2.$$

**Corollary 7.4** *The following bound holds*

$$\|B_1(t)Z_1(t)\|_\beta \leq C \|Z_1(t)\|_{-\beta} \int_t^{t_1} (1 + |d_1(\tau)|)^{2\beta} \|Z_1(\tau)\|_{-\beta}^2 d\tau, \quad 0 \leq t \leq t_1. \quad (7.9)$$

For the proof we apply Lemma 7.2 to (7.5) and to the bounds for all the rest terms of  $B_1(t)$ .

## 8 Integral Inequality

The equation (7.4) can be written in the integral form:

$$Z_1(t) = e^{A_1 t} Z_1(0) + \int_0^t e^{A_1(t-s)} [B_1 Z_1(s) + N_1(s)] ds, \quad 0 \leq t \leq t_1. \quad (8.1)$$

We apply the symplectic orthogonal projection  $\mathbf{P}_1 := \mathbf{P}_{v(t_1)}$  to both sides, and get

$$\mathbf{P}_1 Z_1(t) = e^{A_1 t} \mathbf{P}_1 Z_1(0) + \int_0^t e^{A_1(t-s)} \mathbf{P}_1 [B_1 Z_1(s) + N_1(s)] ds.$$

We have used here that  $\mathbf{P}_1$  commutes with the group  $e^{A_1 t}$  since the space  $\mathcal{Z}_1 := \mathbf{P}_1 \mathcal{E}$  is invariant with respect to  $e^{A_1 t}$  by Proposition 6.8 ii) (cf. also Remark 6.7). Applying (6.28) we obtain that

$$\|\mathbf{P}_1 Z_1(t)\|_{-\beta} \leq \frac{C}{(1+t)^\beta} \|\mathbf{P}_1 Z_1(0)\|_\beta + C \int_0^t \frac{1}{(1+|t-s|)^\beta} \|\mathbf{P}_1 [B_1 Z_1(s) + N_1(s)]\|_\beta ds. \quad (8.2)$$

The operator  $\mathbf{P}_1 = \mathbf{I} - \mathbf{\Pi}_1$  is continuous in  $\mathcal{E}_\beta$  by (6.25). Hence, from (8.2) and (7.8), (7.9), we obtain that

$$\begin{aligned} \|\mathbf{P}_1 Z_1(t)\|_{-\beta} &\leq \frac{C}{(1+t)^\beta} \|Z_1(0)\|_\beta \\ &+ C(\bar{d}_1(t_1)) \int_0^t \frac{1}{(1+|t-s|)^\beta} \left[ \|Z_1(s)\|_{-\beta} \int_s^{t_1} \|Z_1(\tau)\|_{-\beta}^2 d\tau + \|Z_1(s)\|_{-\beta}^2 \right] ds, \quad 0 \leq t \leq t_1, \end{aligned} \quad (8.3)$$

where  $\bar{d}_1(s) := \sup_{0 \leq t \leq s} |d_1(t)|$ ,  $0 \leq s \leq t_1$ . Since  $\|Z_1(t)\|_{-\beta} \leq C(\bar{d}_1)\|Z(t)\|_{-\beta}$  by Lemma 7.2, we can rewrite (8.3) as

$$\begin{aligned} \|\mathbf{P}_1 Z_1(t)\|_{-\beta} &\leq \frac{C(\bar{d}_1)}{(1+t)^\beta} \|Z(0)\|_\beta \\ &+ C(\bar{d}_1) \int_0^t \frac{1}{(1+|t-s|)^\beta} \left[ \|Z(s)\|_{-\beta} \int_s^{t_1} \|Z(\tau)\|_{-\beta}^2 d\tau + \|Z(s)\|_{-\beta}^2 \right] ds, \quad 0 \leq t \leq t_1, \end{aligned} \quad (8.4)$$

We can replace here the constants  $C(\bar{d}_1)$  by  $C$  if  $\bar{d}_1$  is bounded for  $t_1 \geq 0$ . In order to do this replacement, we reduce the exit time

**Definition 8.1**  $t'_*$  is the exit time

$$t'_* = \sup\{t \in [0, t_*) : \bar{d}_1(s) \leq 1, \quad 0 \leq s \leq t\}. \quad (8.5)$$

Now (8.4) implies that for  $t_1 < t'_*$

$$\begin{aligned} \|\mathbf{P}_1 Z_1(t)\|_{-\beta} &\leq \frac{C}{(1+t)^\beta} \|Z(0)\|_\beta \\ &+ C \int_0^t \frac{1}{(1+|t-s|)^\beta} \left[ \|Z(s)\|_{-\beta} \int_s^{t_1} \|Z(\tau)\|_{-\beta}^2 d\tau + \|Z(s)\|_{-\beta}^2 \right] ds, \quad 0 \leq t \leq t_1, \end{aligned} \quad (8.6)$$

## 9 Symplectic Orthogonality

Finally, we are going to change  $\mathbf{P}_1 Z_1(t)$  by  $Z(t)$  in the left hand side of (8.6). We will prove that it is possible using again that  $d_\beta \ll 1$  in (2.9). For the justification we reduce the exit time once more. First, we introduce the “majorant”

$$m(t) := \sup_{s \in [0, t]} (1+s)^\beta \|Z(s)\|_{-\beta}, \quad t \in [0, t_*]. \quad (9.1)$$

Let us denote by  $\varepsilon$  a fixed positive number which we will specify below.

**Definition 9.1**  $t''_*$  is the exit time

$$t''_* = \sup\{t \in [0, t'_*) : m(s) \leq \varepsilon, \quad 0 \leq s \leq t\}. \quad (9.2)$$

The following important bound (9.3) allows us to change the norm of  $\mathbf{P}_1 Z_1(t)$  in the left hand side of (8.6) by the norm of  $Z(t)$ .

**Lemma 9.2** For sufficiently small  $\varepsilon > 0$ , we have for  $t_1 < t''_*$

$$\|Z(t)\|_{-\beta} \leq C \|\mathbf{P}_1 Z_1(t)\|_{-\beta}, \quad 0 \leq t \leq t_1, \quad (9.3)$$

where  $C$  depends only on  $\rho$  and  $\bar{v}$ .

**Proof** The proof is based on the symplectic orthogonality (6.27), i.e.

$$\mathbf{\Pi}_{v(t)}Z(t) = 0, \quad t \in [0, t_1], \quad (9.4)$$

and on the fact that all the spaces  $\mathcal{Z}(t) := \mathbf{P}_{v(t)}\mathcal{E}$  are “almost parallel” for all  $t$ .

Namely, we first note that  $\|Z(t)\|_{-\beta} \leq C\|Z_1(t)\|_{-\beta}$  by Lemma 7.2, since  $|d_1(t)| \leq 1$  for  $t \leq t_1 < t''_* < t'_*$ . Therefore, it suffices to prove that

$$\|Z_1(t)\|_{-\beta} \leq 2\|\mathbf{P}_1Z_1(t)\|_{-\beta}, \quad 0 \leq t \leq t_1. \quad (9.5)$$

This estimate will follow from

$$\|\mathbf{\Pi}_{v(t_1)}Z_1(t)\|_{-\beta} \leq \frac{1}{2}\|Z_1(t)\|_{-\beta}, \quad 0 \leq t \leq t_1, \quad (9.6)$$

since  $\mathbf{P}_1Z_1(t) = Z_1(t) - \mathbf{\Pi}_{v(t_1)}Z_1(t)$ . To prove (9.6), we write (9.4) as

$$\mathbf{\Pi}_{v(t),1}Z_1(t) = 0, \quad t \in [0, t_1], \quad (9.7)$$

where  $\mathbf{\Pi}_{v(t),1}Z_1(t)$  is  $\mathbf{\Pi}_{v(t)}Z(t)$  expressed in terms of the variable  $y_1 = y + d_1(t)$ . Hence, (9.6) follows from (9.7) if the difference  $\mathbf{\Pi}_{v(t_1)} - \mathbf{\Pi}_{v(t),1}$  is small uniformly in  $t$ , i.e.

$$\|\mathbf{\Pi}_{v(t_1)} - \mathbf{\Pi}_{v(t),1}\| < 1/2, \quad 0 \leq t \leq t_1. \quad (9.8)$$

It remains to justify (9.8) for a sufficiently small  $\varepsilon > 0$ .

In order to prove the bound (9.8), we will need the formula (6.25) and the following relation which follows from (6.25):

$$\mathbf{\Pi}_{v(t),1}Z_1(t) = \sum \mathbf{\Pi}_{jl}(v(t))\tau_{j,1}(v(t))\Omega(\tau_{l,1}(v(t)), Z_1(t)), \quad (9.9)$$

where  $\tau_{j,1}(v(t))$  are the vectors  $\tau_j(v(t))$  expressed in the variables  $y_1$ . In detail (cf. (3.4)),

$$\begin{aligned} \tau_{j,1}(v) &:= (-\partial_j E_v(y_1 - d_1(t)), -\partial_j A_v(y_1 - d_1(t)), e_j, 0), \\ \tau_{j+3,1}(v) &:= (\partial_{v_j} E_v(y_1 - d_1(t)), \partial_{v_j} A_v(y_1 - d_1(t)), 0, \partial_{v_j} P_v), \end{aligned} \quad \left| \quad j = 1, 2, 3, \right. \quad (9.10)$$

where  $v = v(t)$ . Since  $|d_1(t)| \leq 1$  and  $\nabla\tau_j$  are smooth and sufficiently fast decaying at infinity functions, Lemma 7.2 implies that

$$\|\tau_{j,1}(v(t)) - \tau_j(v(t))\|_{\beta} \leq C|d_1(t)|^{\beta}, \quad 0 \leq t \leq t_1 \quad (9.11)$$

for all  $j = 1, 2, \dots, 6$ . Furthermore,

$$\tau_j(v(t)) - \tau_j(v(t_1)) = \int_t^{t_1} \dot{v}(s) \cdot \nabla_v \tau_j(v(s)) ds,$$

and therefore

$$\|\tau_j(v(t)) - \tau_j(v(t_1))\|_{\beta} \leq C \int_t^{t_1} |\dot{v}(s)| ds, \quad 0 \leq t \leq t_1. \quad (9.12)$$

Similarly,

$$|\mathbf{\Pi}_{jl}(v(t)) - \mathbf{\Pi}_{jl}(v(t_1))| = \left| \int_t^{t_1} \dot{v}(s) \cdot \nabla_v \mathbf{\Pi}_{jl}(v(s)) ds \right| \leq C \int_t^{t_1} |\dot{v}(s)| ds, \quad 0 \leq t \leq t_1, \quad (9.13)$$

since  $|\nabla_v \mathbf{\Pi}_{jl}(v(s))|$  is uniformly bounded by (6.9). Hence, the bounds (9.8) will follow from (6.25), (9.9) and (9.11)-(9.13) if we establish that  $|d_1(t)|$  and the integral in the right hand side of (9.12) can be made as small as we please by choosing  $\varepsilon > 0$  sufficiently small.

To estimate  $d_1(t)$ , we note that

$$w(s) - v(t_1) = w(s) - v(s) + v(s) - v(t_1) = \dot{c}(s) + \int_s^{t_1} \dot{v}(\tau) d\tau \quad (9.14)$$

by (6.10). Hence, (7.2), Lemma 6.2 and the definition (9.1) imply that

$$\begin{aligned} |d_1(t)| &= \left| \int_{t_1}^t (w(s) - v(t_1)) ds \right| \leq \int_t^{t_1} \left( |\dot{c}(s)| + \int_s^{t_1} |\dot{v}(\tau)| d\tau \right) ds \\ &\leq C m^2(t_1) \int_t^{t_1} \left( \frac{1}{(1+s)^{2\beta}} + \int_s^{t_1} \frac{d\tau}{(1+\tau)^{2\beta}} \right) ds \leq C m^2(t_1) \leq C \varepsilon^2, \quad 0 \leq t \leq t_1 \end{aligned} \quad (9.15)$$

since  $t_1 < t''_*$ . Similarly,

$$\int_t^{t_1} |\dot{v}(s)| ds \leq C m^2(t_1) \int_t^{t_1} \frac{ds}{(1+s)^{2\beta}} \leq C \varepsilon^2, \quad 0 \leq t \leq t_1. \quad (9.16)$$

The proof is completed.  $\square$

## 10 Decay of Transversal Component

Here we prove Proposition 6.4.

*Step i)* We fix  $\varepsilon > 0$  and  $t''_* = t''_*(\varepsilon)$  for which Lemma 9.2 holds. Then the bound of type (8.6) holds with  $\|\mathbf{P}_1 Z_1(t)\|_{-\beta}$  in the left hand side replaced by  $\|Z(t)\|_{-\beta}$  :

$$\begin{aligned} \|Z(t)\|_{-\beta} &\leq \frac{C}{(1+t)^\beta} \|Z(0)\|_\beta \\ &+ C \int_0^t \frac{1}{(1+|t-s|)^\beta} \left[ \|Z(s)\|_{-\beta} \int_s^{t_1} \|Z(\tau)\|_{-\beta}^2 d\tau + \|Z(s)\|_{-\beta}^2 \right] ds, \quad 0 \leq t \leq t_1 \end{aligned} \quad (10.1)$$

for  $t_1 < t'_*$ . This implies an integral inequality for the majorant

$$m(t) := \sup_{s \in [0, t]} (1+s)^\beta \|Z(s)\|_{-\beta}.$$

Namely, multiplying both sides of (10.1) by  $(1+t)^\beta$ , and taking the supremum in  $t \in [0, t_1]$ , we get

$$m(t_1) \leq C \|Z(0)\|_\beta + C \sup_{t \in [0, t_1]} \int_0^t \frac{(1+t)^\beta}{(1+|t-s|)^\beta} \left[ \frac{m(s)}{(1+s)^\beta} \int_s^{t_1} \frac{m^2(\tau) d\tau}{(1+\tau)^{2\beta}} + \frac{m^2(s)}{(1+s)^{2\beta}} \right] ds$$

for  $t_1 \leq t''_*$ . Taking into account that  $m(t)$  is a monotone increasing function, we get

$$m(t_1) \leq C \|Z(0)\|_\beta + C [m^3(t_1) + m^2(t_1)] I(t_1), \quad t_1 \leq t''_*. \quad (10.2)$$

where

$$I(t_1) = \sup_{t \in [0, t_1]} \int_0^t \frac{(1+t)^\beta}{(1+|t-s|)^\beta} \left[ \frac{1}{(1+s)^\beta} \int_s^{t_1} \frac{d\tau}{(1+\tau)^{2\beta}} + \frac{1}{(1+s)^{2\beta}} \right] ds \leq \bar{I} < \infty, \quad t_1 \geq 0.$$

Therefore, (10.2) becomes

$$m(t_1) \leq C \|Z(0)\|_\beta + C \bar{I} [m^3(t_1) + m^2(t_1)], \quad t_1 < t''_*. \quad (10.3)$$

This inequality implies that  $m(t_1)$  is bounded for  $t_1 < t''_*$ , and moreover,

$$m(t_1) \leq C_1 \|Z(0)\|_\beta, \quad t_1 < t''_*, \quad (10.4)$$

since  $m(0) = \|Z(0)\|_\beta$  is sufficiently small by (3.7).

*Step ii)* The constant  $C_1$  in the estimate (10.4) does not depend on  $t_*$ ,  $t'_*$  and  $t''_*$  by Lemma 9.2. We choose  $d_\beta$  in (2.9) so small that  $\|Z(0)\|_\beta < \varepsilon/(2C_1)$ . It is possible due to (3.7). Then the estimate (10.4) implies that  $t''_* = t'_*$  and therefore (10.4) holds for all  $t_1 < t'_*$ . Then the bound (9.15) holds for all  $t < t'_*$ . We choose  $\varepsilon$  so small that the right hand side in (9.15) does not exceed one. Then  $t'_* = t_*$ . Therefore, (10.4) holds for all  $t_1 < t_*$ , hence (6.4) also holds if  $\|Z(0)\|_\beta$  is sufficiently small. Finally, this implies that  $t_* = \infty$ , hence also  $t''_* = t'_* = \infty$  and (10.4) holds for all  $t_1 > 0$  if  $d_\beta$  is small enough.  $\square$

## 11 Soliton Asymptotics

Here we prove our main Theorem 2.2 under the assumption that the decay (6.19) holds. First we will prove the asymptotics (2.10) for the vector components, and afterwards the asymptotics (2.11) for the fields.

**Asymptotics for the vector components** From (4.2) we have  $\dot{q} = \dot{b} + \dot{r}$ , and from (6.21), (6.22), (4.15) it follows that  $\dot{r} = -B_{v(t)}\langle \rho, a \rangle + B_{v(t)}\pi + \mathcal{O}(\|Z\|_{-\beta}^2)$ . Thus,

$$\dot{q} = \dot{b} + \dot{r} = v(t) + \dot{c}(t) - B_{v(t)}\langle \rho, a \rangle + B_{v(t)}\pi + \mathcal{O}(\|Z\|_{-\beta}^2). \quad (11.1)$$

Recall that  $\beta = 1 + \delta$ ,  $0 < \delta < 1/2$ . The equation (6.11) and the estimates (6.12), (6.19) imply that

$$|\dot{c}(t)| + |\dot{v}(t)| \leq \frac{C_1(\rho, \bar{v}, d_\beta)}{(1+t)^{2+2\delta}}, \quad t \geq 0. \quad (11.2)$$

Therefore,  $c(t) = c_+ + \mathcal{O}(t^{-(1+2\delta)})$  and  $v(t) = v_+ + \mathcal{O}(t^{-(1+2\delta)})$ ,  $t \rightarrow \infty$ . Since  $\|a\|_{-1,\beta}$  and  $|\pi|$  decay like  $(1+t)^{-\beta}$ , the estimate (6.19), and (11.2), (11.1) imply that

$$\dot{q}(t) = v_+ + \mathcal{O}(t^{-\beta}). \quad (11.3)$$

Similarly,

$$b(t) = c(t) + \int_0^t v(s)ds = v_+t + a_+ + \mathcal{O}(t^{-2\delta}), \quad (11.4)$$

hence the second part of (2.10) follows:

$$q(t) = b(t) + r(t) = v_+t + a_+ + \mathcal{O}(t^{-2\delta}), \quad (11.5)$$

since  $r(t) = \mathcal{O}(t^{-\beta})$  by (6.19).

**Asymptotics for the fields** We apply the approach developed in [18], see also [15, 17, 19, 24]. For the field part of the solution,  $F(t) = (E(x, t), A(x, t))$  let us define the accompanying soliton field as  $F_{v(t)}(t) = (E_{v(t)}(x - q(t)), A_{v(t)}(x - q(t)))$ , where we define now  $v(t) = \dot{q}(t)$ , cf. (11.1). Then for the difference  $Z(t) = F(t) - F_{v(t)}(t)$  we obtain easily the equation [24], Eq. (2.5),

$$\dot{Z}(t) = \mathcal{A}Z(t) - \dot{v} \cdot \nabla_v F_{v(t)}(t), \quad \mathcal{A}(E, A) = (-\Delta A, -E).$$

Then

$$Z(t) = W^0(t)Z(0) - \int_0^t W^0(t-s)[\dot{v}(s) \cdot \nabla_v F_{v(s)}(s)]ds. \quad (11.6)$$

To obtain the asymptotics (2.11) it suffices to prove that  $Z(t) = W^0(t)\Psi_+ + r_+(t)$  with some  $\Psi_+ \in \mathcal{F}$  and  $\|r_+(t)\|_{\mathcal{F}} = \mathcal{O}(t^{-\delta})$ . This is equivalent to

$$W^0(-t)Z(t) = \Psi_+ + r'_+(t), \quad (11.7)$$

where  $\|r'_+(t)\|_{\mathcal{F}} = \mathcal{O}(t^{-\delta})$  since  $W^0(t)$  is a unitary group in the Sobolev space  $\mathcal{F}$  by the energy conservation for the free wave equation. Finally, (11.7) holds since (11.6) implies that

$$W^0(-t)Z(t) = Z(0) + \int_0^t W^0(-s)R(s)ds, \quad R(s) = \dot{v}(s) \cdot \nabla_v F_{v(s)}(s), \quad (11.8)$$

where the integral in the right hand side of (11.8) converges in the Hilbert space  $\mathcal{F}$  with the rate  $\mathcal{O}(t^{-\delta})$ . The latter holds since  $\|W^0(-s)R(s)\|_{\mathcal{F}} = \mathcal{O}(s^{-\beta})$  by the unitarity of  $W^0(-s)$  and the decay rate  $\|R(s)\|_{\mathcal{F}} = \mathcal{O}(s^{-\beta})$  which follows from the asymptotics for the vector components. More precisely, differentiating the equation (1.3) in  $t$  and using the asymptotics (11.3), (6.19) we obtain an estimate for  $\dot{v}(t) = \ddot{q}(t)$  providing the mentioned decay rate of  $R(s)$ .  $\square$

## 12 Decay for the Linearized Dynamics

In remaining sections we prove Proposition 6.8 in order to complete the proof of the main result (Theorem 2.2). Here we discuss our general strategy of the proof of the Proposition. We apply the Fourier-Laplace transform

$$\tilde{X}(\lambda) = \int_0^\infty e^{-\lambda t} X(t) dt, \quad \operatorname{Re} \lambda > 0 \quad (12.1)$$

to (6.23). According to Proposition 6.8, we expect that the solution  $X(t)$  is bounded in the norm  $\|\cdot\|_{-\beta}$ . Then the integral (12.1) converges and is analytic for  $\operatorname{Re} \lambda > 0$ , and

$$\|\tilde{X}(\lambda)\|_{-\beta} \leq \frac{C}{\operatorname{Re} \lambda}, \quad \operatorname{Re} \lambda > 0. \quad (12.2)$$

Let us derive an equation for  $\tilde{X}(\lambda)$  which is equivalent to the Cauchy problem for (6.23) with the initial condition  $X(0) = X_0 \in \mathcal{E}_{-\beta}$ . We will write  $A$  and  $v$  instead of  $A_1$  and  $v_1$  in all remaining part of the paper. Applying the Fourier-Laplace transform to (6.23), we get that

$$\lambda \tilde{X}(\lambda) = A \tilde{X}(\lambda) + X_0, \quad \operatorname{Re} \lambda > 0. \quad (12.3)$$

Let us stress that (12.3) is equivalent to the Cauchy problem for the functions  $X(t) \in C[0, \infty; \mathcal{E}_{-\beta}]$ . Hence the solution  $X(t)$  is given by

$$\tilde{X}(\lambda) = -(A - \lambda)^{-1} X_0, \quad \operatorname{Re} \lambda > 0 \quad (12.4)$$

if the resolvent  $R(\lambda) = (A - \lambda)^{-1}$  exists for  $\operatorname{Re} \lambda > 0$ .

Let us comment on our following strategy in proving the decay (6.19). First, we will construct the resolvent  $R(\lambda)$  for  $\operatorname{Re} \lambda > 0$  and prove that it is a continuous operator in  $\mathcal{E}_{-\beta}$ . Then  $\tilde{X}(\lambda) \in \mathcal{E}_{-\beta}$  and is an analytic function for  $\operatorname{Re} \lambda > 0$ . Second, we have to justify that there exist a (unique) function  $X(t) \in C[0, \infty; \mathcal{E}_{-\beta}]$  satisfying (12.1).

The analyticity of  $\tilde{X}(\lambda)$  and Paley-Wiener arguments (see [23]) should provide the existence of a  $\mathcal{E}_{-\beta}$ -valued distribution  $X(t)$ ,  $t \in \mathbb{R}$ , with a support in  $[0, \infty)$ . Formally,

$$X(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\omega t} \tilde{X}(i\omega + 0) d\omega, \quad t \in \mathbb{R}. \quad (12.5)$$

However, to check the continuity of  $X(t)$  for  $t \geq 0$ , we need additionally a bound for  $\tilde{X}(i\omega + 0)$  at large  $|\omega|$ . Finally, for the time decay of  $X(t)$ , we need an additional information on the smoothness and decay of  $\tilde{X}(i\omega + 0)$ . More precisely, we should prove that the function  $\tilde{X}(i\omega + 0)$

- i) is smooth for  $\omega \neq 0$ ,
- ii) decays in a certain sense as  $|\omega| \rightarrow \infty$ .

iii) is analytic at  $\omega = 0$  if  $X_0 \in \mathcal{Z}_v := \mathbf{P}_v \mathcal{E}$  and  $X_0 \in \mathcal{E}_\beta$ .

Then the decay (6.19) would follow from the Fourier-Laplace representation (12.5).

However, we will check the properties of type i)-iii) only for the last two components  $\tilde{r}(\lambda)$  and  $\tilde{\pi}(\lambda)$  of the vector  $\tilde{X}(\lambda) = (\tilde{e}(\lambda), \tilde{a}(\lambda), \tilde{r}(\lambda), \tilde{\pi}(\lambda))$ . The properties provide the decay (6.19) for the vector components  $r(t)$  and  $\pi(t)$  of the solution  $X(t)$ .

At the end, we analyze the field components  $e(x, t)$  and  $a(x, t)$  without the Fourier-Laplace transform. Namely, we prove the decay (6.19) for the field components directly from the field equations of the system (6.23), using the decay of the components  $r(t), \pi(t)$  and a version of strong Huygens principle for the Maxwell equation.

## 13 Constructing the Resolvent

First let us make a change of variables in the equation (6.23) which will simplify significant the structure of the resolvent. The equation (6.23) reads

$$\begin{aligned} \dot{e} &= v \cdot \nabla e - \Delta a + \Pi_s(r \cdot \nabla \rho v - \rho B_v(\pi - \langle \rho, a \rangle)), \\ \dot{a} &= -e + v \cdot \nabla a, \\ \dot{r} &= B_v(\pi - \langle \rho, a \rangle), \\ \dot{\pi} &= \langle \rho, v \cdot \nabla a \rangle - \langle r \cdot \nabla \rho, v \cdot \nabla A_v \rangle. \end{aligned} \quad (13.1)$$

Put  $\varphi = \pi - \langle \rho, a \rangle$ . Then  $\pi = \varphi + \langle \rho, a \rangle$ . If we prove a decay of  $\varphi$  and  $a$ , then  $\pi$  has the corresponding decay as well. Further

$$\dot{\varphi} = \dot{\pi} - \langle \rho, \dot{a} \rangle = \dot{\pi} - \langle \rho, -e + v \cdot \nabla a \rangle = \langle \rho, e \rangle - \langle r \cdot \nabla \rho, v \cdot \nabla A_v \rangle$$

by the last equation of (13.1). Thus, the system (13.1) is equivalent to the following system,

$$\begin{aligned} \dot{e} &= v \cdot \nabla e - \Delta a + \Pi_s(r \cdot \nabla \rho v - \rho B_v \varphi), \\ \dot{a} &= -e + v \cdot \nabla a, \\ \dot{r} &= B_v \varphi, \\ \dot{\varphi} &= \langle \rho, e \rangle - \langle r \cdot \nabla \rho, v \cdot \nabla A_v \rangle. \end{aligned} \quad (13.2)$$

Denote by the same letter  $A$  the operator

$$A \begin{pmatrix} e \\ a \\ r \\ \varphi \end{pmatrix} := \begin{pmatrix} v \cdot \nabla e - \Delta a + \Pi_s(r \cdot \nabla \rho v - \rho B_v \varphi) \\ -e + v \cdot \nabla a \\ B_v \varphi, \\ \langle \rho, e \rangle - \langle r \cdot \nabla \rho, v \cdot \nabla A_v \rangle \end{pmatrix}. \quad (13.3)$$

Below we construct and study the resolvent of  $A$ .

To justify the representation (12.4), we construct the resolvent as a bounded operator in  $\mathcal{E}_{-\beta}$  for  $\text{Re } \lambda > 0$ . We shall write  $(e(y), a(y), r, \varphi)$  instead of  $(\tilde{e}(y, \lambda), \tilde{a}(y, \lambda), \tilde{r}(\lambda), \tilde{\varphi}(\lambda))$  to simplify the notations. Then (12.3) reads

$$(A - \lambda) \begin{pmatrix} e \\ a \\ r \\ \varphi \end{pmatrix} = - \begin{pmatrix} e_0 \\ a_0 \\ r_0 \\ \varphi \end{pmatrix}.$$



This gives the system of equations

$$\left. \begin{aligned} v \cdot \nabla e - \Delta a + \Pi_s(r \cdot \nabla \rho v - \rho B_v \varphi) - \lambda e &= -e_0 \\ -e + v \cdot \nabla a - \lambda a &= -a_0 \\ B_v \varphi - \lambda r &= -r_0 \\ \langle \rho, e \rangle - \langle r \cdot \nabla \rho, v \cdot \nabla A_v \rangle - \lambda \varphi &= -\varphi_0 \end{aligned} \right| \quad (13.4)$$

*Step i)* Let us consider the first two equations. In the Fourier space they become

$$\left. \begin{aligned} -i(kv)\hat{e} + k^2\hat{a} - \hat{\Pi}_s(i(kr)\hat{\rho}v + \hat{\rho}B_v\varphi) - \lambda\hat{e} &= -\hat{e}_0 \\ -\hat{e} - i(kv)\hat{a} - \lambda\hat{a} &= -\hat{a}_0 \end{aligned} \right| \quad (13.5)$$

From the last equation we have

$$\hat{e} = -(\lambda + i(kv))\hat{a} + \hat{a}_0. \quad (13.6)$$

Substitute to the first equation of (13.5) and obtain

$$\hat{a} = \frac{1}{\hat{D}}((\lambda + i(kv))\hat{a}_0 - \hat{e}_0 + \hat{\Pi}), \quad \hat{\Pi} := \hat{\rho}\hat{\Pi}_s(i(kr)v + B_v\varphi), \quad (13.7)$$

where

$$\hat{D} = \hat{D}(\lambda) = k^2 + (\lambda + i(kv))^2. \quad (13.8)$$

It is easy to see that

$$\hat{D}(\lambda) \neq 0 \text{ for } \operatorname{Re} \lambda > 0. \quad (13.9)$$

Finally,

$$\hat{e} = \frac{k^2\hat{a}_0 + (\lambda + i(kv))\hat{e}_0 - (\lambda + i(kv))\hat{\Pi}}{\hat{D}}. \quad (13.10)$$

Let us proceed to the fourth equations of (13.4). The equation reads

$$\langle \rho, e \rangle - \langle r \cdot \nabla \rho, v \cdot \nabla A_v \rangle - \lambda \varphi = -\varphi_0.$$

From now on we use the system of coordinates in  $x$ -space in which  $v = (|v|, 0, 0)$ , hence  $vk = |v|k_1$ . By (13.10) and a straightforward computation we obtain

$$\langle \rho, e \rangle = \Phi + Cr + F\varphi,$$

where

$$\Phi = \Phi(\lambda, \rho, a_0, e_0) := \int \frac{(k^2\hat{a}_0 + (\lambda + i(kv))\hat{e}_0)\bar{\hat{\rho}}}{\hat{D}} dk.$$

**Remark 13.1** *Note that*

$$\Phi = \tilde{F}_{t \rightarrow \lambda} \langle W^1(t)(e_0, a_0), \rho \rangle, \quad (13.11)$$

where  $\tilde{F}_{t \rightarrow \lambda}$  is Laplace transform in  $t$ ,  $W^1(t)$  is the first component of the dynamical group  $W(t)$  defined below by (18.5).

Further,

$$C = i|v| \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}, \quad c_1 = - \int \frac{k_1(\lambda + ik_1|v|)|\hat{\rho}|^2}{\hat{D}(\lambda)} (1 - \frac{k_1^2}{k^2}) dk, \quad (13.12)$$

$$c_j = \int \frac{k_1(\lambda + ik_1|v|)|\hat{\rho}|^2 k_j^2}{\hat{D}(\lambda) k^2} dk, \quad j = 2, 3, \quad (13.13)$$

$$F = \begin{pmatrix} \nu^3 f_1 & 0 & 0 \\ 0 & \nu f_2 & 0 \\ 0 & 0 & \nu f_3 \end{pmatrix}, \quad f_j = \int \frac{(\lambda + ik_1|v|)|\hat{\rho}|^2}{\hat{D}(\lambda)} (\frac{k_j^2}{k^2} - 1) dk, \quad j = 1, 2, 3; \quad (13.14)$$

recall that  $\nu = \sqrt{1 - v^2}$ . Further,  $-\langle r \cdot \nabla \rho, v \cdot \nabla A_v \rangle = Gr$ , where

$$G = v^2 \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix}, \quad g_1 = \int \frac{k_1^2 |\hat{\rho}|^2}{k^2 - k_1^2 v^2} (\frac{k_1^2}{k^2} - 1) dk, \quad g_j = \int \frac{k_1^2 k_j^2 |\hat{\rho}|^2}{k^2 (k^2 - k_1^2 v^2)} dk, \quad j = 2, 3. \quad (13.15)$$

By the change of variables  $k_2 \mapsto k_3$  we obtain that  $c_2 = c_3$ ,  $f_2 = f_3$ ,  $g_2 = g_3$ . Let us denote  $c := c_2 = c_3$ ,  $f := f_2 = f_3$ ,  $g := g_2 = g_3$ . Note that  $c_1 + c_2 + c_3 = 0$  and thus,  $c_1 = -c_2 - c_3 = -2c$ . Similarly,  $g_1 = -2g$ , and the matrices  $C$  and  $G$  simplify to

$$C = i|v| \begin{pmatrix} -2c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix}, \quad G = v^2 \begin{pmatrix} -2g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{pmatrix}. \quad (13.16)$$

Finally, the fourth equation becomes

$$(C + G)r + (F - \lambda E)\varphi = -(\varphi_0 + \Phi). \quad (13.17)$$

We write the third equation of (13.4) and the equation (13.17) together in the form

$$M(\lambda) \begin{pmatrix} r \\ \varphi \end{pmatrix} = \begin{pmatrix} -r_0 \\ -(\varphi_0 + \Phi) \end{pmatrix}, \quad (13.18)$$

where  $M(\lambda)$  is the  $6 \times 6$ -matrix

$$M(\lambda) = \begin{pmatrix} -\lambda E & B_v \\ C + G & F - \lambda E \end{pmatrix}. \quad (13.19)$$

Assume for a moment that the matrix  $M(\lambda)$  is invertible for  $\text{Re } \lambda > 0$  (later we shall prove that this the case indeed). Then

$$\begin{pmatrix} r \\ \varphi \end{pmatrix} = M^{-1}(\lambda) \begin{pmatrix} -r_0 \\ -(\varphi_0 + \Phi) \end{pmatrix}, \quad \text{Re } \lambda > 0. \quad (13.20)$$

Finally, formulas (13.7), (13.10), and (13.20) give the expression of the resolvent  $R(\lambda) = (A - \lambda)^{-1}$ ,  $\text{Re } \lambda > 0$ , in Fourier representation.

*Step ii)* Let us now proceed to the  $x$ -representation. We invert the matrix of the system (13.5) and obtain

$$\begin{pmatrix} -(ivk + \lambda) & k^2 \\ -1 & -(ivk + \lambda) \end{pmatrix}^{-1} = [(ivk + \lambda)^2 + k^2]^{-1} \begin{pmatrix} -(ivk + \lambda) & -k^2 \\ 1 & -(ivk + \lambda) \end{pmatrix}.$$

Taking the inverse Fourier transform, we obtain the corresponding fundamental solution

$$G_\lambda(y) = \begin{pmatrix} v \cdot \nabla - \lambda & \Delta \\ 1 & v \cdot \nabla - \lambda \end{pmatrix} g_\lambda(y), \quad (13.21)$$

where  $g_\lambda(y)$  is the unique tempered fundamental solution of the determinant

$$D = D(\lambda) = -\Delta + (-v \cdot \nabla + \lambda)^2. \quad (13.22)$$

Thus,

$$g_\lambda(y) = F_{k \rightarrow y}^{-1} \frac{1}{k^2 + (ivk + \lambda)^2} = F_{k \rightarrow y}^{-1} \frac{1}{k^2 + (i|v|k_1 + \lambda)^2}, \quad y \in \mathbb{R}^3. \quad (13.23)$$

Note that the denominator does not vanish for  $\operatorname{Re} \lambda > 0$ . This implies

**Lemma 13.2** *The operator  $G_\lambda$  with the integral kernel  $G_\lambda(y - y')$  is continuous as an operator from  $H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$  to  $H^2(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3)$  for  $\operatorname{Re} \lambda > 0$ .*

Thus, formulas (13.5) and (13.21) imply the convolution representation

$$\begin{aligned} e &= -(v \cdot \nabla - \lambda)g_\lambda * e_0 - \Delta g_\lambda * a_0 + (v \cdot \nabla - \lambda)g_\lambda * \Pi \\ a &= -g_\lambda * e_0 - (v \cdot \nabla - \lambda)g_\lambda * e_0 + g_\lambda * \Pi \end{aligned} \quad (13.24)$$

*Step iii)* Let us compute  $g_\lambda(y)$  explicitly. First consider the case  $v = 0$ . The fundamental solution of the operator  $-\Delta + \lambda^2$  is

$$g_\lambda(y) = \frac{e^{-\lambda|y|}}{4\pi|y|}. \quad (13.25)$$

Thus, in the case  $v = 0$  we have

$$G_\lambda(y - y') = \begin{pmatrix} -\lambda & \Delta \\ 1 & -\lambda \end{pmatrix} \frac{e^{-\lambda|y-y'|}}{4\pi|y-y'|}.$$

For general  $v = (|v|, 0, 0)$  with  $|v| < 1$  the denominator in (13.23), which is the Fourier symbol of  $D$ , reads

$$\begin{aligned} \hat{D}(k) &= k^2 + (i|v|k_1 + \lambda)^2 = (1 - v^2)k_1^2 + k_2^2 + k_3^2 + 2i|v|k_1\lambda + \lambda^2 \\ &= (1 - v^2)\left(k_1 + \frac{i|v|\lambda}{1 - v^2}\right)^2 + k_2^2 + k_3^2 + \frac{\lambda^2}{1 - v^2}. \end{aligned} \quad (13.26)$$

Set  $\gamma = (1 - v^2)^{-1/2}$  and return to  $x$ -space:

$$D = -\frac{1}{\gamma^2}(\nabla_1 + \gamma^2|v|\lambda)^2 - \nabla_2^2 - \nabla_3^2 + (\gamma\lambda)^2. \quad (13.27)$$

Define  $\tilde{y}_1 := \gamma y_1$  and  $\tilde{\nabla}_1 := \partial/\partial\tilde{y}_1$ . Then

$$D = -(\tilde{\nabla}_1 + \gamma|v|\lambda)^2 - \nabla_2^2 - \nabla_3^2 + (\gamma\lambda)^2. \quad (13.28)$$

Thus, its fundamental solution is

$$g_\lambda(y) = \frac{e^{-\gamma\lambda|\tilde{y}| - \gamma|v|\lambda\tilde{y}_1}}{4\pi|\tilde{y}|}, \quad \tilde{y} := (\gamma y_1, y_2, y_3), \quad (13.29)$$

where  $\operatorname{Re} \lambda > 0$ . Let us note that

$$0 < \operatorname{Re} \gamma|v|\lambda < \operatorname{Re} \gamma\lambda, \quad (13.30)$$

since  $|v| < 1$ .

*Step iv)* Let us state the result which we have got above.

**Lemma 13.3** *i) The operator  $D = D(\lambda)$  is invertible in  $L^2(\mathbb{R}^3)$  for  $\operatorname{Re} \lambda > 0$  and its fundamental solution (13.29) decays exponentially.*

*ii) Formula (13.29) implies that, for every fixed  $y$ , the Green function  $g_\lambda(y)$  admits an analytic continuation (in the variable  $\lambda$ ) to the entire complex plane.*

**Lemma 13.4** *The matrix function  $M(\lambda)$  ( $M^{-1}(\lambda)$ ) admits an analytic (meromorphic) continuation from the domain  $\operatorname{Re} \lambda > 0$  to the entire complex plane.*

**Proof** The analytic continuation of  $M(\lambda)$  exists by Lemma 13.3 ii) and the expressions (13.12)-(13.14) since the function  $\rho(x)$  is compactly supported by (1.12). The inverse matrix is then meromorphic since it exists for large  $\operatorname{Re} \lambda$ : this follows from (13.19) since  $C(\lambda), F(\lambda) \rightarrow 0$  as  $\operatorname{Re} \lambda \rightarrow \infty$  by (13.12)-(13.14).  $\square$

## 14 Analiticity for $\operatorname{Re} \lambda > 0$

**Proposition 14.1** *The operator-valued function  $R(\lambda) : \mathcal{E} \rightarrow \mathcal{E}$  is analytic for  $\operatorname{Re} \lambda > 0$ .*

**Proof** It is sufficient to prove that the operator  $A - \lambda : \mathcal{E} \rightarrow \mathcal{E}$  has a bounded inverse operator for  $\operatorname{Re} \lambda > 0$ . Let us recall, that  $A = A_{v,v}$  where  $|v| < 1$ .

*Step i)* Let us prove that  $\operatorname{Ker}(A - \lambda) = 0$  for  $\operatorname{Re} \lambda > 0$ . Indeed, let us assume that  $X_\lambda = (e, a, r, \varphi) \in \mathcal{E}$  satisfies  $(A - \lambda)X_\lambda = 0$ , that is  $X_\lambda$  is a solution to (13.4) with  $e_0 = a_0 = 0$  and  $r_0 = \varphi_0 = 0$ . We have to prove that  $X_\lambda = 0$ .

First let us check that  $\varphi_\lambda = 0$ . Indeed, the trajectory  $X := X_\lambda e^{\lambda t} \in C(\mathbb{R}, \mathcal{E})$  is the solution to the equation  $\dot{X} = AX$  that is (5.13) with  $w = v$ . Then  $\mathcal{H}_{v,v}(X(t))$  grows exponentially by (5.19), since the matrix  $B_v$  is positive. This growth contradicts to the conservation of  $\mathcal{H}_{v,v}$ . Now  $\lambda r_\lambda = B_v \varphi_\lambda = 0$  by the third equation of (13.4), hence  $r_\lambda = 0$  since  $\lambda \neq 0$ . Finally,  $e_\lambda = 0, a_\lambda = 0$  by the equations (13.24) with  $r = r_\lambda = 0$ .

*Step ii)* One has

$$(A - \lambda) \begin{pmatrix} e \\ a \\ r \\ \varphi \end{pmatrix} = \begin{pmatrix} v \cdot \nabla - \lambda & -\Delta & \Pi_s(\cdot \nabla \rho v) & \Pi_s \rho B_v \\ -1 & v \cdot \nabla - \lambda & 0 & 0 \\ 0 & 0 & -\lambda & B_v \\ \langle \rho, \cdot \rangle & 0 & \langle \nabla \rho, v \cdot A_v \rangle & -\lambda \end{pmatrix} \begin{pmatrix} e \\ a \\ r \\ \varphi \end{pmatrix}.$$

Thus,  $A - \lambda = A_0 + T$ , where

$$A_0 = \begin{pmatrix} v \cdot \nabla - \lambda & -\Delta & 0 & 0 \\ -1 & v \cdot \nabla - \lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & \Pi_s(\cdot \nabla \rho v) & \Pi_s \rho B_v \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_v \\ \langle \rho, \cdot \rangle & 0 & \langle \nabla \rho, v \cdot A_v \rangle & 0 \end{pmatrix}.$$

The operator  $T$  is finite-dimensional, and the operator  $A_0^{-1}$  is bounded in  $\mathcal{E}$  by Lemma 13.3. Finally,  $A - \lambda = A_0(I + A_0^{-1}T)$ , where  $A_0^{-1}T$  is a compact operator. Since we know that  $\operatorname{Ker}(I + A_0^{-1}T) = 0$ , the operator  $(I + A_0^{-1}T)$  is invertible by Fredholm theory.  $\square$

**Corollary 14.2** *The matrix  $M(\lambda)$  of (13.19) is invertible for  $\operatorname{Re} \lambda > 0$ .*

## 15 Regularity on the Imaginary Axis

Next step should be an investigation of the limit values of the resolvent  $R(\lambda)$  at the imaginary axis  $\lambda = i\omega, \omega \in \mathbb{R}$ , that is necessary for proving the decay (6.19) of the solution  $X(t) = (e(t), a(t), r(t), \varphi(t))$ .

However, we consider only the limit values of the matrix  $M^{-1}(\lambda)$ , inverse to (13.19). This allows us to prove the decay of the finite dimensional components  $r(t), \varphi(t)$  by the inverse Fourier-Laplace transform. We will derive the decay of the field components  $e(t), a(t)$  later without the Fourier-Laplace transform.

First, let us describe the continuous spectrum of the operator  $A = A_{v,v}$  on the imaginary axis. By definition, the continuous spectrum corresponds to  $\omega \in \mathbb{R}$ , such that the resolvent  $R(i\omega + 0)$  is not a bounded operator in  $\mathcal{E}$ . By the formulas (13.24), this is the case when the Green function  $g_\lambda(y - y')$  loses the exponential decay. This is equivalent to the condition that  $\operatorname{Re} \varkappa = 0$  where  $\varkappa = \gamma\sqrt{-\omega^2}$ . Thus,  $i\omega$  belongs to the continuous spectrum for all  $\omega \in \mathbb{R}$ . By Lemma 13.4, the limit matrix

$$M(i\omega) := M(i\omega + 0) = \begin{pmatrix} -i\omega E & B_v \\ C(i\omega + 0) + G & F(i\omega + 0) - i\omega E \end{pmatrix}, \quad \omega \in \mathbb{R}, \quad (15.1)$$

exists, and its entries are continuous functions of  $\omega \in \mathbb{R}$ , smooth for  $|\omega| > 0$ . Recall that the point  $\lambda = 0$  belongs to the discrete spectrum of the operator  $A$  by Lemma 5.3 i), hence  $M(i\omega + 0)$  (probably) is also not invertible at  $\omega = 0$ .

**Proposition 15.1** *Let  $\rho$  satisfy the Wiener condition (1.13), and  $|v| < 1$ . Then the limit matrix  $M(i\omega + 0)$  is invertible for  $\omega \neq 0$ ,  $\omega \in \mathbb{R}$ .*

**Proof** Since  $v = (|v|, 0, 0)$ , the matrix  $B_v$  is also diagonal:

$$B_v := \nu(E - v \otimes v) = \begin{pmatrix} \nu^3 & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \nu \end{pmatrix}. \quad (15.2)$$

By (15.1) and (13.12) – (13.16), for  $\omega \in \mathbb{R}$ ,

$$\det M(i\omega) = \det \begin{pmatrix} -i\omega E & B_v \\ C(i\omega + 0) + G & F(i\omega + 0) - i\omega E \end{pmatrix} = D_1 D^2,$$

where

$$D_1 = -\omega^2 - i\omega\nu^3 f_1(i\omega + 0) - \nu^3 i|v|c_1(i\omega + 0) - \nu^3 v^2 g_1, \quad (15.3)$$

$$D = -\omega^2 - i\omega\nu f(i\omega + 0) - \nu i|v|c(i\omega + 0) - \nu v^2 g. \quad (15.4)$$

The formula for the determinant is obvious since all of the matrices  $C$ ,  $F$ ,  $G$ , and  $B_v$  are diagonal.

For  $|\omega| > 0$  the invertibility of  $M(i\omega)$  follows from (15.3), (15.4) by the following lemma.

**Lemma 15.2** *If (1.13) holds and  $|\omega| > 0$ , then the imaginary parts of  $D_1$  and  $D$  are positive:  $\operatorname{Im} D_1 > 0$ ,  $\operatorname{Im} D > 0$ .*

**Proof** Let  $\omega > 0$ , the case  $\omega < 0$  is similar. Note that

$$\operatorname{Im} D_1 = -\nu^3(\omega \operatorname{Re} f_1(i\omega + 0) + |v| \operatorname{Re} c_1(i\omega + 0)).$$

For  $\varepsilon > 0$  we have

$$f_j(i\omega + \varepsilon) = \int \frac{(i\omega + \varepsilon + ik_1|v|)|\hat{\rho}(k)|^2}{\hat{D}(i\omega + \varepsilon, k)} \left(\frac{k_j^2}{k^2} - 1\right) dk, \quad j = 1, 2, 3. \quad (15.5)$$

By Plemelj formula for  $C^1$ -functions, cf. [40] (see Chapter VII, formula (58)) and [20],

$$\operatorname{Re} f_j(i\omega + 0) = \pi \int_{T_\omega} \frac{(\omega + k_1|v|)|\hat{\rho}(k)|^2}{|\nabla \hat{D}(i\omega, k)|} \left(\frac{k_j^2}{k^2} - 1\right) dS, \quad (15.6)$$

where

$$T_\omega = \{k : (\nu k_1 - \frac{|v|\omega}{\nu})^2 + k_2^2 + k_3^2 = \frac{\omega^2}{\nu^2}\}$$

is the ellipsoid on which  $\hat{D}(i\omega, k) = 0$ . Similarly,

$$\operatorname{Re} c_1(i\omega + 0) = \pi \int_{T_\omega} \frac{k_1(\omega + k_1|v|)|\hat{\rho}(k)|^2}{|\nabla \hat{D}(i\omega, k)|} \left(\frac{k_1^2}{k^2} - 1\right) dS. \quad (15.7)$$

Then

$$\omega \operatorname{Re} f_1(i\omega + 0) + |v| \operatorname{Re} c_1(i\omega + 0) = \pi \int_{T_\omega} \frac{(\omega + k_1|v|)^2 |\hat{\rho}(k)|^2}{|\nabla \hat{D}(i\omega, k)|} \left(\frac{k_1^2}{k^2} - 1\right) dS.$$

By the Wiener condition (1.13) we obtain that the last integral is negative and hence  $\operatorname{Im} D_1 > 0$ . Further,

$$\operatorname{Im} D = -\nu(\omega \operatorname{Re} f(i\omega + 0) + |v| \operatorname{Re} c(i\omega + 0)).$$

Recall that  $f := f_2 = f_3$  and by (13.14) we obtain that

$$f(i\omega + \varepsilon) = \int \frac{(i\omega + \varepsilon + ik_1|v|)|\hat{\rho}(k)|^2}{\hat{D}(i\omega + \varepsilon, k)} \left(\frac{k_2^2 + k_3^2}{2k^2} - 1\right) dk.$$

Then

$$\begin{aligned} \operatorname{Re} f(i\omega + 0) &= \pi \int_{T_\omega} \frac{(\omega + k_1|v|)|\hat{\rho}(k)|^2}{|\nabla \hat{D}(i\omega, k)|} \left(\frac{k_2^2 + k_3^2}{2k^2} - 1\right) dS = \\ &\pi \int_{T_\omega} \frac{(\omega + k_1|v|)|\hat{\rho}(k)|^2}{|\nabla \hat{D}(i\omega, k)|} \left(\frac{(\omega + k_1|v|)^2 - k_1^2}{2k^2} - 1\right) dS, \end{aligned}$$

since  $k_2^2 + k_3^2 = (\omega + k_1|v|)^2 - k_1^2$  on  $T_\omega$ . Now recall that  $c := c_2 = c_3$  and  $c_1 = -2c$ , by (13.12) we obtain that

$$c(i\omega + \varepsilon) = -\frac{1}{2} \int \frac{k_1(i\omega + \varepsilon + ik_1|v|)|\hat{\rho}(k)|^2}{\hat{D}(i\omega + \varepsilon, k)} \left(\frac{k_1^2}{k^2} - 1\right) dk.$$

Then

$$\operatorname{Re} c(i\omega + 0) = -\frac{\pi}{2} \int_{T_\omega} \frac{k_1(\omega + k_1|v|)|\hat{\rho}(k)|^2}{|\nabla \hat{D}(i\omega, k)|} \left(\frac{k_1^2}{k^2} - 1\right) dS.$$

We take into account that  $k^2 = (\omega + k_1|v|)^2$  on  $T_\omega$  and obtain  $\omega \operatorname{Re} f(i\omega + 0) + |v| \operatorname{Re} c(i\omega + 0) =$

$$-\frac{\pi}{2} \int_{T_\omega} \frac{|\hat{\rho}(k)|^2 (\omega + k_1|v|)^2 (\omega^2 + k_1^2(1 - v^2))}{|\nabla \hat{D}(i\omega, k)| k^2} dk.$$

This value is negative, again by the Wiener condition, hence  $\operatorname{Im} D > 0$ . This completes the proofs of the lemma and the Proposition 15.1.  $\square$

**Corollary 15.3** *Proposition 15.1 implies that the matrix  $M^{-1}(i\omega)$  is smooth in  $\omega \in \mathbb{R} \setminus 0$ .*

**Remark 15.4** The proof of the Lemma 15.2 is the unique point in the paper where the Wiener condition is indispensable.

## 16 Singular Spectral Points

Let us recall that the formula (13.20) expresses the Fourier-Laplace transforms  $\tilde{r}(\lambda), \tilde{\varphi}(\lambda)$ . Hence, the components are given by the Fourier integral

$$\begin{pmatrix} r(t) \\ \varphi(t) \end{pmatrix} = \frac{1}{2\pi} \int e^{i\omega t} M^{-1}(i\omega + 0) \begin{pmatrix} -r_0 \\ -(\varphi_0 + \Phi) \end{pmatrix} d\omega \quad (16.1)$$

if it converges in the sense of distributions. The Corollary 15.3 alone is not sufficient for the proof of the convergence and decay of the integral. Namely, we need an additional information about a regularity of the matrix  $M^{-1}(i\omega)$  at its singular point  $\omega = 0$  and some bounds at  $|\omega| \rightarrow \infty$ . We will analyze all the points separately.

**I.** First we study the asymptotic behavior of  $M^{-1}(\lambda)$  at infinity. Let us recall that  $M^{-1}(\lambda)$  was originally defined for  $\operatorname{Re} \lambda > 0$ , but it admits a meromorphic continuation to the entire complex plane  $\mathbb{C}$  (see Lemma 13.4).

The following Lemma is a very particular case of a general fundamental theorem about the bound for the truncated resolvent on the continuous spectrum. The bound plays a crucial role in the study of the long-time asymptotics of general linear hyperbolic PDEs, [40].

**Proposition 16.1** *There exist a matrix  $R_0$  and a matrix-function  $R_1(\omega)$ , such that*

$$M^{-1}(i\omega) = \frac{R_0}{\omega} + R_1(\omega), \quad |\omega| \geq 1, \quad \omega \in \mathbb{R},$$

where, for every  $k = 0, 1, 2, \dots$ ,

$$|\partial_\omega^k R_1(\omega)| \leq \frac{C_k}{|\omega|^2}, \quad |\omega| \geq 1, \quad \omega \in \mathbb{R}. \quad (16.2)$$

The proof is similar to that of Proposition 16.2 in [20], see also [38, Thm 3], [1, the bound (A.2')], [22, Thm 8.1], [39, Thm 3]).

**II.** Finally, we consider the point  $\omega = 0$  which is most singular. The point is an isolated pole of a finite degree by Lemma 13.4, hence the Laurent expansion holds,

$$M^{-1}(i\omega) = \sum_{k=0}^n L_k \omega^{-k-1} + h(\omega), \quad |\omega| < \varepsilon_0, \quad (16.3)$$

where  $L_k$  are  $6 \times 6$  complex matrices,  $\varepsilon_0 > 0$ , and  $h(\omega)$  is an analytic matrix-valued function for complex  $\omega$  with  $|\omega| < \varepsilon_0$ .

## 17 Time Decay of the Vector Components

Here we prove the decay (6.19) for the components  $r(t)$  and  $\varphi(t)$ .

**Lemma 17.1** *Let  $X_0 \in \mathcal{Z}_{v,\beta}$ . Then  $Q(t), P(t)$  are continuous and*

$$|r(t)| + |\varphi(t)| \leq \frac{C(\rho, \bar{v}, d_{-\beta})}{(1 + |t|)^\beta}, \quad t \geq 0. \quad (17.4)$$

**Proof** The expansions (16.2) and (16.3) imply the convergence of the Fourier integral (16.1) in the sense of distributions to a continuous function of  $t \geq 0$ . Let us prove (17.4). First let us note that the condition  $X_0 \in \mathcal{Z}_{v,\beta}$  implies that the whole trajectory  $X(t)$  lies in  $\mathcal{Z}_{v,\beta}$ . This follows from the invariance of the space  $\mathcal{Z}_{v,\beta}$  under the generator  $A_{v,v}$  (cf. Remark 6.7). Note that for  $X_0$  not belonging to  $\mathcal{Z}_{v,\beta}$

the components  $Q(t)$  and  $P(t)$  may contain non-decaying terms which correspond to the singular point  $\omega = 0$ . Indeed, we know that the linearized dynamics admits the secular solutions without decay, see (6.24). The formulas (3.4) give the corresponding components  $r_S(t)$  and  $\varphi_S(t)$  of the secular solutions,

$$\begin{pmatrix} r_S(t) \\ \varphi_S(t) \end{pmatrix} = \sum_1^3 C_j \left[ \begin{pmatrix} e_j \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ \partial_{v_j} P_v \end{pmatrix} \right] + \sum_1^3 D_j \begin{pmatrix} e_j \\ 0 \end{pmatrix}. \quad (17.5)$$

We will show that the symplectic orthogonality condition leads to (17.4). Let us split the Fourier integral (16.1) into two terms using the partition of unity  $\zeta_1(\omega) + \zeta_2(\omega) = 1$ ,  $\omega \in \mathbb{R}$ :

$$\begin{aligned} \begin{pmatrix} r(t) \\ \varphi(t) \end{pmatrix} &= \frac{1}{2\pi} \int e^{i\omega t} (\zeta_1(\omega) + \zeta_2(\omega)) M^{-1}(i\omega + 0) \begin{pmatrix} -r_0 \\ -(\varphi_0 + \Phi) \end{pmatrix} d\omega \\ &= I_1(t) + I_2(t), \end{aligned} \quad (17.6)$$

where the functions  $\zeta_k(\omega) \in C^\infty(\mathbb{R})$  are supported by

$$\left. \begin{aligned} \text{supp } \zeta_1 &\subset \{ \omega \in \mathbb{R} : \varepsilon_0/2 < |\omega| \} \\ \text{supp } \zeta_2 &\subset \{ \omega \in \mathbb{R} : |\omega| < \varepsilon_0 \} \end{aligned} \right\} \quad (17.7)$$

Then

i) The function  $I_1(t)$  decays like of  $(1 + |t|)^{-\beta}$  due to Proposition 16.1.

First let us prove the continuity. Consider the contribution of

$$\frac{\zeta_1}{\omega} R_0 \begin{pmatrix} -r_0 \\ -\varphi_0 \end{pmatrix} = \left( \frac{\zeta_1}{\omega} - \text{V.P.} \frac{1}{\omega} + \text{V.P.} \frac{1}{\omega} \right) R_0 \begin{pmatrix} -r_0 \\ -\varphi_0 \end{pmatrix}.$$

Here  $\frac{\zeta_1}{\omega} - \text{V.P.} \frac{1}{\omega}$  forms the compactly supported distribution which inverse Laplace transform is smooth. The inverse Laplace transform of  $\text{V.P.} \frac{1}{\omega}$  is  $\text{sgn } t$  which is continuous for  $t \geq 0$  (in the sense of distributions). The contribution of

$$\zeta_1 R_1(\omega) \begin{pmatrix} -r_0 \\ -\varphi_0 \end{pmatrix}$$

is continuous, since  $\zeta_1 R_1(\omega)$  is summable by Proposition 16.1. Further, consider the contribution of

$$\frac{\zeta_1}{\omega} R_0 \begin{pmatrix} 0 \\ \tilde{F}_{t \rightarrow \omega} \langle W^1(t)(e_0, a_0), \rho \rangle \end{pmatrix}.$$

Note that  $\tilde{F}_{\omega \rightarrow t}^{-1} \frac{\zeta_1}{\omega} R_0$  is continuous by the previous argument and has the decay  $t^{-\infty}$ . The function  $\langle W^1(t)(e_0, a_0), \rho \rangle$  is continuous and decays like  $t^{-\beta}$  by Lemma 18.3. Thus, we come to the convolution of two continuous functions with decays  $t^{-\infty}$  and  $t^{-\beta}$ . This convolution is clearly continuous. The similar argument is applied to the term

$$\zeta_1 R_1(\omega) \begin{pmatrix} 0 \\ \tilde{F}_{t \rightarrow \omega} \langle W^1(t)(e_0, a_0), \rho \rangle \end{pmatrix}.$$

The continuity is proved.

Now let us prove the decay like  $t^{-\beta}$ . We need to consider only the contribution of the term

$$\zeta_1 \left( \frac{1}{\omega} R_0 + R_1(\omega) \right) \begin{pmatrix} 0 \\ \tilde{F}_{t \rightarrow \omega} \langle W^1(t)(e_0, a_0), \rho \rangle \end{pmatrix}.$$



All the rest terms give the decay  $t^{-\infty}$ : differentiate any times in  $\omega$  and obtain a summable function of  $\omega$ . Consider

$$\zeta_1 \frac{1}{\omega} R_0 \left( \begin{array}{c} 0 \\ \tilde{F}_{t \rightarrow \omega} \langle W^1(t)(e_0, a_0), \rho \rangle \end{array} \right).$$

The inverse Laplace transform gives the convolution of functions with decays  $t^{-\infty}$  and  $t^{-\beta}$ . This convolution decays like  $t^{-\beta}$ . The last term can be considered similarly.

ii) The function  $I_2(t)$  decays like  $t^{-\infty}$  if  $(e_0, a_0, r_0, \varphi_0) \in \mathcal{Z}_{\beta, v}$ .

We claim that the operator  $A|_{\mathcal{Z}_{\beta, v}}$  is invertible (recall that  $\mathcal{Z}_{\beta, v}$  is invariant w.r.t.  $A$ ). First let us show that it has the trivial kernel. We have to resolve the system (13.4) with  $(e_0, a_0, r_0, \varphi_0) = 0$  and check that  $(e, a, r, \varphi) = 0$  if  $Y = (e, a, r, \varphi + \langle a, \rho \rangle)$  satisfies the symplectic orthogonality conditions. The vector part of the system, (13.18) becomes

$$\begin{pmatrix} 0 & B_v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r \\ \varphi \end{pmatrix} = 0,$$

hence  $\varphi = 0$ . The equations (13.7), (13.10) give  $\hat{a} = i\hat{\rho}\hat{\Pi}_s(kr)v/\hat{D}$ ,  $\hat{e} = -ikv\hat{\rho}\hat{\Pi}_s(kr)v/\hat{D}$ . For  $j = 1, 2, 3$  the symplectic orthogonality conditions  $\Omega(Y, \tau_j) = 0$  are satisfied automatically and imply no restriction on  $r$ . The symplectic orthogonality conditions  $\Omega(Y, \tau_{j+3}) = 0$  transform to  $Mr = 0$ , where  $M$  is a positive matrix (see the details in Appendix C), and thus,  $r = 0$ . This imply  $a = 0$ ,  $e = 0$  and the triviality of the kernel is proved.

Further,  $A$  acts in the two invariant subspaces: in  $\mathcal{Z}_{\beta, v}$  and in its 6D symplectic orthogonal complement; and the index of  $A$  is zero since it is an elliptic operator. Consider a homotopy which takes  $A$  to zero on this 6D subspace. Then the index of  $A|_{\mathcal{Z}_{\beta, v}}$  remains zero and  $A|_{\mathcal{Z}_{\beta, v}}$  has the trivial kernel. Then it has the trivial co-kernel and thus is invertible.

Thus,  $R(\lambda)|_{\mathcal{Z}_{\beta, v}}$  is analytic for  $\lambda = i\omega$ ,  $|\omega| < \varepsilon_0$ . Now,

$$M^{-1}(i\omega + 0) \begin{pmatrix} -r_0 \\ -(\varphi_0 + \Phi) \end{pmatrix}$$

is the vector component of  $R(\lambda)|_{\mathcal{Z}_{\beta, v}}(Z_0)$  and hence is analytic in  $\lambda$  or  $\omega$ . Then the  $I_2(t)$  decays like  $t^{-\infty}$ .  $\square$

## 18 Time Decay of Fields

Here we construct the field components  $e(x, t), a(x, t)$  of the solution  $X(t)$  and prove their decay corresponding to (6.19). Let us denote  $F(t) = (e(\cdot, t), a(\cdot, t))$ . We will construct the fields solving the first two equations of (6.23), where  $A$  is given by (4.15). These two equations have the form

$$\dot{F}(t) = \begin{pmatrix} v \cdot \nabla & -\Delta \\ -1 & v \cdot \nabla \end{pmatrix} F + \begin{pmatrix} \Pi(t) \\ 0 \end{pmatrix}, \quad \Pi(t) := \Pi_s(r(t)\nabla\rho v - \rho B_v\varphi(t)). \quad (18.1)$$

By Lemma 17.1 we know that  $r(t)$  and  $\varphi(t)$  are continuous and

$$|r(t)| + |\varphi(t)| \leq \frac{C(\rho, \bar{v}, d_{-\beta})}{(1 + |t|)^\beta}, \quad t \geq 0. \quad (18.2)$$

Hence, the Proposition 6.8 is reduced now to the following

**Proposition 18.1** *i) Let functions  $r(t), \varphi(t) \in C[0, \infty; \mathbb{R}^3]$ , and  $F_0 \in \mathcal{F}$ . Then the equation (18.1) admits a unique solution  $F(t) \in C[0, \infty; \mathcal{F})$  with the initial condition  $F(0) = F_0$ .*

*ii) If  $F_0 \in \mathcal{F}_\beta$  and the decay (18.2) holds, the corresponding fields also decay uniformly in  $v$ :*

$$\|F(t)\|_{-\beta} \leq \frac{C(\rho, \bar{v}, d_{-\beta})}{(1 + |t|)^\beta} \|F_0\|_\beta, \quad t \geq 0, \quad (18.3)$$

for  $|v| \leq \bar{v}$  with any  $\bar{v} \in (0, 1)$ .

**Proof** The statement i) follows from the Duhamel representation

$$F(t) = W(t)F_0 + \left[ \int_0^t W(t-s) \begin{pmatrix} \Pi(s) \\ 0 \end{pmatrix} ds \right], \quad t \geq 0, \quad (18.4)$$

where  $W(t)$  is the dynamical group of the modified wave equation

$$\dot{F}(t) = \begin{pmatrix} v \cdot \nabla & -\Delta \\ -1 & v \cdot \nabla \end{pmatrix} F(t). \quad (18.5)$$

The group  $W(t)$  can be expressed through the group  $W_0(t)$  of the wave equation

$$\dot{\Phi}(t) = \begin{pmatrix} 0 & -\Delta \\ -1 & 0 \end{pmatrix} \Phi(t). \quad (18.6)$$

Namely, the problem (18.6) corresponds to (18.5), when  $v = 0$ , and it is easy to see that

$$[W(t)F(0)](x) = [W_0(t)F(0)](x + vt), \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}. \quad (18.7)$$

The identity (18.7) implies the energy conservation law for the group  $W(t)$ . Namely, for  $(e(\cdot, t), a(\cdot, t)) = W(t)F(0)$  we have

$$\int (|e(x, t) - v \cdot \nabla a(x, t)|^2 + |\nabla a(x, t)|^2) dx = \text{const}, \quad t \in \mathbb{R}.$$

In particular, this gives that

$$\|W(t)F_0\|_{\mathcal{F}} \leq C(\bar{v})\|F_0\|_{\mathcal{F}}, \quad t \in \mathbb{I}. \quad (18.8)$$

This estimate and (18.4) imply the statement i).

Let us proceed to the statement ii). *Step i)*

**Lemma 18.2** *For any  $F_0 \in \mathcal{F}_\beta$  the following decay holds,*

$$\|W_0(t)F_0\|_{-\beta} \leq \frac{C(\beta)}{(1+t)^\beta} \|F_0\|_\beta, \quad t \geq 0, \quad (18.9)$$

*for the dynamical group  $W_0(t)$  corresponding to the wave equation (18.6).*

For proof see Appendix B.

*Step ii)*

**Lemma 18.3** *For  $\bar{v} < 1$  and  $F_0 \in \mathcal{F}_\beta$ , the following decay holds,*

$$\|W(t)F_0\|_{-\beta} \leq \frac{C(\beta, \bar{v})}{(1+t)^\beta} \|F_0\|_\beta, \quad t \geq 0, \quad (18.10)$$

*for the dynamical group  $W(t)$  corresponding to the modified wave equation (18.5) with  $|v| < \bar{v}$ .*

**Proof** Let  $(e(x, t), a(x, t)) = [W(t)F_0](x)$ ,  $(\tilde{e}(x, t), \tilde{a}(x, t)) = [W_0(t)F_0](x)$ . Since  $[W(t)F_0](x) = [W_0(t)F_0](x + vt)$ , we have  $e(x, t) = \tilde{e}(x + vt, t)$ ,  $a(x, t) = \tilde{a}(x + vt, t)$ . The norms  $\|\tilde{a}(\cdot, t)\|_{1, -\beta}$  and  $\|\tilde{e}(\cdot, t)\|_{-\beta}$  are bounded by  $C(\beta)\|F_0\|_\beta(1+|t|)^{-\beta}$ . Let us start with  $e$ :

$$\begin{aligned} \int (1+|x|)^{-2\beta} |e|^2(x, t) dx &= \int (1+|x|)^{-2\beta} |\tilde{e}|^2(x + vt, t) dx = \int (1+|y - vt|)^{-2\beta} |\tilde{e}|^2(y, t) dy \leq \\ &C(\bar{v}) \int (1+|y|)^{-2\beta} |\tilde{e}|^2(y, t) dy, \end{aligned}$$

since  $|v| \leq \bar{v} < 1$ . The last integral is bounded by  $C(\beta)\|F_0\|_\beta^2(1+|t|)^{-2\beta}$ . The norm  $\|a\|_{1,-\beta} = \|(1+|x|)^{-\beta}\nabla a\|_{L^2}$  can be estimated similarly.

*Step iii)* Now the statement ii) follows from the lemma 18.3 and the Duhamel representation (18.4). Indeed,  $W(t)F_0$  decays like  $t^{-\beta}$  by lemma 18.3. Further,

$$\begin{aligned} \left\| \int_0^t W(t-s) \begin{pmatrix} \Pi(s) \\ 0 \end{pmatrix} ds \right\|_{-\beta} &\leq \int_0^t \|W(t-s) \begin{pmatrix} \Pi(s) \\ 0 \end{pmatrix}\|_{-\beta} ds \leq \\ &\leq C \int_0^t \frac{\|\Pi(s)\|_{-\beta} ds}{(1+(t-s))^\beta} \leq C' \int_0^t \frac{ds}{(1+(t-s))^\beta(1+s)^\beta} \end{aligned}$$

by Lemma 18.3 and (18.2). The last integral decays like  $(1+t)^{-\beta}$  by a well-known result on decay of a convolution.  $\square$

## A Appendix: Computing Symplectic Form

Here we compute the matrix elements  $\Omega(\tau_j, \tau_l)$  of the matrix  $\Omega$  and prove that the matrix is non-degenerate. For  $j, l = 1, 2, 3$  it follows from (3.4) and (3.2) that

$$\Omega(\tau_j, \tau_l) = \langle \partial_j E_v, \partial_l A_v \rangle - \langle \partial_j A_v, \partial_l E_v \rangle, \quad (\text{A.1})$$

$$\Omega(\tau_{j+3}, \tau_{l+3}) = \langle \partial_{v_j} E_v, \partial_{v_l} A_v \rangle - \langle \partial_{v_j} A_v, \partial_{v_l} E_v \rangle, \quad (\text{A.2})$$

and

$$\Omega(\tau_j, \tau_{l+3}) = -\langle \partial_j E_v, \partial_{v_l} A_v \rangle + \langle \partial_j A_v, \partial_{v_l} E_v \rangle + e_j \cdot \partial_{v_l} P_v. \quad (\text{A.3})$$

In Fourier representation the solitons read

$$\hat{E}_v(k) = \frac{i(kv)\hat{\rho}}{D} \left( \frac{(kv)}{k^2}k - v \right), \quad \hat{A}_v(k) = \frac{-\hat{\rho}}{D} \left( \frac{(kv)}{k^2}k - v \right), \quad (\text{A.4})$$

$$P_v = p_v + \langle A_v, \rho \rangle = p_v + v \int \frac{|\hat{\rho}|^2 dk}{D} - \int \frac{|\hat{\rho}|^2 dk}{k^2 D} (kv)k, \quad (\text{A.5})$$

where  $D := k^2 - (kv)^2$ ;  $D$  is nonnegative and even in  $k$ . Differentiating in  $v$  we obtain for  $j = 1, 2, 3$ :

$$\partial_{v_j} \hat{E}_v = \frac{i\hat{\rho}}{D} \left( \frac{2k_j(kv)}{D}k - \frac{k_j(k^2 + (kv)^2)}{D}v - (kv)e_j \right), \quad (\text{A.6})$$

$$\partial_{v_l} \hat{A}_v = \frac{\hat{\rho}}{D} \left( \frac{2k_l(kv)}{D}v - \frac{k_l(k^2 + (kv)^2)}{k^2 D}k + e_l \right), \quad (\text{A.7})$$

$$\partial_{v_l} P_v = \partial_{v_l} p_v + \langle \partial_{v_l} A_v, \rho \rangle =$$

$$B_v^{-1}e_l + \int \frac{|\hat{\rho}|^2 dk}{D} e_l + 2 \int \frac{|\hat{\rho}|^2 (kv)k_l dk}{D^2} v - \int \frac{|\hat{\rho}|^2 (k^2 + (kv)^2)k_l dk}{k^2 D^2} k. \quad (\text{A.8})$$

Then for  $j, l = 1, 2, 3$  we get from (A.1) by the Parseval identity,

$$\langle \partial_j E_v, \partial_l A_v \rangle = -i \int \frac{k_j k_l (kv) |\hat{\rho}|^2}{D^2} \left( \frac{(kv)}{k^2}k - v \right)^2 dk = 0,$$

since the integrand function is odd in  $k$ . Similarly,  $\langle \partial_j A_v, \partial_l E_v \rangle = 0$  and thus

$$\Omega(\tau_j, \tau_l) = 0. \quad (\text{A.9})$$

Further, by (A.2),

$$\begin{aligned} \langle \partial_{v_j} E_v, \partial_{v_l} A_v \rangle = i \int \frac{|\hat{\rho}|^2}{D^2} & \left( \frac{4k_j k_l (kv)^3}{D^2} + \frac{2k_j k_l (kv)}{D} - \right. \\ & \frac{2k_j k_l (kv)(k^2 + (kv)^2)}{D^2} - \frac{2k_j k_l (kv)(k^2 + (kv)^2)v^2}{D^2} - \frac{k_j(k^2 + (kv)^2)v_l}{D} + \\ & \left. \frac{k_j k_l (kv)(k^2 + (kv)^2)^2}{k^2 D^2} - \frac{2k_l (kv)^2 v_j}{D} - (kv)\delta_{jl} + \frac{k_j k_l (kv)(k^2 + (kv)^2)}{k^2 D} \right) dk = 0, \end{aligned}$$

since the integrand function is odd in  $k$ . Note that the integral converges by the neutrality condition (1.14). Similarly,  $\langle \partial_{v_j} A_v, \partial_{v_l} E_v \rangle = 0$  and thus,

$$\Omega(\tau_{j+3}, \tau_{l+3}) = 0. \quad (\text{A.10})$$

Now let us compute  $\Omega(\tau_j, \tau_{l+3})$ . First,

$$-\langle \partial_j E_v, \partial_{v_l} A_v \rangle = \int \frac{k_j (kv) |\hat{\rho}|^2 dk}{D^2} \left( v_l + \frac{2k_l (kv) v^2}{D} - \frac{k_l (kv)(k^2 + (kv)^2)}{k^2 D} \right).$$

Second,

$$\langle \partial_j A_v, \partial_{v_l} E_v \rangle = \int \frac{k_j |\hat{\rho}|^2 dk}{D^2} \left( (kv)v_l + \frac{k_l(k^2 + (kv)^2)v^2}{D} - \frac{2k_l (kv)^2}{D} \right).$$

And third,

$$e_j \cdot \partial_{v_l} P_v = e_j \cdot B_v^{-1} e_l + \int \frac{|\hat{\rho}|^2 dk}{D} \delta_{jl} + 2 \int \frac{|\hat{\rho}|^2 (kv) k_l dk}{D^2} v_j - \int \frac{|\hat{\rho}|^2 (k^2 + (kv)^2) k_j k_l dk}{k^2 D^2}.$$

We use the rotational invariance of  $\rho$  and apply the orthogonal change of variables  $k = V k'$ ,  $v = V v'$ , where  $v' = (|v|, 0, 0)$  and  $V = \|V_{jl}\|$  is an orthogonal matrix with  $V_{j1} = v_j/|v|$ . Then

$$\int \frac{|\hat{\rho}|^2 (kv) k_l dk}{D^2} = \int \frac{|\hat{\rho}|^2 (k' v') (V k')_l dk'}{(k'^2 - (k' v')^2)^2} = \int \frac{|\hat{\rho}|^2 (k'_1 |v|) (V k')_l dk'}{(k'^2 - k_1'^2 v^2)^2}.$$

In the expression  $k'_1 |v| (V_{l1} k'_1 + V_{l2} k'_2 + V_{l3} k'_3)$  we keep only the term  $k_1'^2 |v| V_{l1} = k_1'^2 v_l$ , since all the rest integrals equal zero. Finally, the integral equals

$$\int \frac{k_1'^2 |\hat{\rho}|^2 dk'}{D'^2} v_l$$

(here  $D' = k'^2 - k_1'^2 v^2$  in the new coordinates) and

$$e_j \cdot \partial_{v_l} P_v = e_j \cdot B_v^{-1} e_l + \int \frac{|\hat{\rho}|^2 dk}{D} \delta_{jl} + 2 \int \frac{k_1'^2 |\hat{\rho}|^2 dk'}{D'^2} v_j v_l - \int \frac{|\hat{\rho}|^2 k_j k_l dk}{D^2} - \int \frac{|\hat{\rho}|^2 (kv)^2 k_j k_l dk}{k^2 D^2}.$$

In total,

$$\begin{aligned} \Omega(\tau_j, \tau_{l+3}) = & e_j \cdot B_v^{-1} e_l + \int \frac{|\hat{\rho}|^2 dk}{D} \delta_{jl} + 4 \int \frac{k_1'^2 |\hat{\rho}|^2 dk'}{D'^2} v_j v_l \\ & + v^2 \int \frac{|\hat{\rho}|^2 k^2 k_j k_l dk}{D^3} - \int \frac{|\hat{\rho}|^2 (kv)^4 k_j k_l dk}{k^2 D^3} \\ & - 3(1 - v^2) \int \frac{|\hat{\rho}|^2 (kv)^2 k_j k_l dk}{D^3} - \int \frac{|\hat{\rho}|^2 k_j k_l dk}{D^2} - \int \frac{|\hat{\rho}|^2 (kv)^2 k_j k_l dk}{k^2 D^2}. \end{aligned} \quad (\text{A.11})$$

The matrix elements of the first line are those of a diagonal positive and nonnegative matrices. In the second line we have

$$\int \frac{|\hat{\rho}|^2(v^2k^4 - (kv)^4)k_jk_l dk}{k^2 D^3}.$$

Since  $v^2k^4 - (kv)^4 \geq 0$ , this is the matrix element  $k_jk_l$  of the nonnegative matrix  $k \otimes k$  integrated with a nonnegative weight, and thus the result is the matrix element of a nonnegative matrix.

Observe that in the third line the matrix element  $k_jk_l$  is integrated every time with a non-positive weight. We apply the described change of variables to all of these integrals. Then we may omit the prime in all of them. We keep only the terms that lead to nonzero integrals and represent the most essential part of the change as the following chain of transforms:

$$k_jk_l \mapsto (V_{j1}k_1 + V_{j2}k_2 + V_{j3}k_3)(V_{l1}k_1 + V_{l2}k_2 + V_{l3}k_3) \mapsto V_{j1}V_{l1}k_1^2 + V_{j2}V_{l2}k_2^2 + V_{j3}V_{l3}k_3^2,$$

since all the rest terms lead to zero integrals. Since the integrals with  $k_2^2$  and  $k_3^2$  are equal by the corresponding change of variables,

$$V_{j1}V_{l1}k_1^2 + V_{j2}V_{l2}k_2^2 + V_{j3}V_{l3}k_3^2 \mapsto V_{j1}V_{l1}k_1^2 + (V_{j2}V_{l2} + V_{j3}V_{l3})k_2^2 \mapsto$$

$$V_{j1}V_{l1}k_1^2 + (V_{j2}V_{l2} + V_{j3}V_{l3} + V_{j1}V_{l1})k_2^2 - V_{j1}V_{l1}k_2^2 = V_{j1}V_{l1}(k_1^2 - k_2^2)$$

by the orthogonality of the matrix  $V$ . Further,

$$V_{j1}V_{l1}(k_1^2 - k_2^2) = \frac{v_jv_l}{v^2}(k_1^2 - k_2^2) \mapsto \frac{v_jv_l}{v^2}(k_1^2 - \frac{k_2^2 + k_3^2}{2}) = \frac{v_jv_l}{v^2}(k_1^2 - \frac{k_1^2 + k_2^2 + k_3^2 - k_1^2}{2}) = \frac{v_jv_l}{v^2} \frac{3k_1^2 - k^2}{2}.$$

Then

$$\begin{aligned} \int \frac{|\hat{\rho}|^2(kv)^2k_jk_l dk}{D^3} &\mapsto \frac{v_jv_l}{2} \int \frac{|\hat{\rho}|^2k_1^2(3k_1^2 - k^2) dk}{D^3}, \\ \int \frac{|\hat{\rho}|^2k_jk_l dk}{D^2} &\mapsto \frac{v_jv_l}{2v^2} \int \frac{|\hat{\rho}|^2(3k_1^2 - k^2) dk}{D^2}, \\ \int \frac{|\hat{\rho}|^2(kv)^2k_jk_l dk}{k^2 D^2} &\mapsto \frac{v_jv_l}{2} \int \frac{|\hat{\rho}|^2k_1^2(3k_1^2 - k^2) dk}{k^2 D^2}. \end{aligned}$$

Finally, in the formula (A.11) for  $\Omega(\tau_j, \tau_{l+3})$  we obtain  $v_jv_l$  multiplied by the following expression:

$$\left(\frac{9}{2} - \frac{3}{2v^2}\right)I_1 + \frac{1}{2v^2}I_2 + \frac{3(1-v^2)}{2}I_3 - \frac{9(1-v^2)}{2}I_4 - \frac{3}{2}I_5, \quad (\text{A.12})$$

where

$$I_1 = \int \frac{|\hat{\rho}|^2k_1^2}{D^2}, \quad I_2 = \int \frac{|\hat{\rho}|^2k^2}{D^2}, \quad I_3 = \int \frac{|\hat{\rho}|^2k^2k_1^2}{D^3}, \quad I_4 = \int \frac{|\hat{\rho}|^2k_1^4}{D^3}, \quad I_5 = \int \frac{|\hat{\rho}|^2k_1^4}{k^2 D^2}.$$

All of the integrals  $I_1$  to  $I_5$  can be computed in terms of  $|v|$  (in further we write  $v$  instead, for simplicity of notation) and  $R := \int_0^{+\infty} |\hat{\rho}|^2 dr$  by the spherical change of variables. After the straightforward computation we get

$$\begin{aligned} I_1 &= \frac{\pi R}{v^3} \left( \frac{1}{1-v} - \frac{1}{1+v} + \log \frac{1-v}{1+v} \right), \quad I_2 = \frac{\pi R}{v} \left( \log \frac{1+v}{1-v} + \frac{1}{1-v} - \frac{1}{1+v} \right), \\ I_3 &= \frac{\pi R}{4v^3} \left( \log \frac{1-v}{1+v} + \frac{1}{1+v} - \frac{1}{1-v} + \frac{1}{(1-v)^2} - \frac{1}{(1+v)^2} \right), \\ I_4 &= \frac{\pi R}{4v^5} \left( 3 \log \frac{1+v}{1-v} + \frac{5}{1+v} - \frac{5}{1-v} + \frac{1}{(1-v)^2} - \frac{1}{(1+v)^2} \right), \end{aligned}$$

$$I_5 = \frac{4\pi R}{v^4} + \frac{\pi R}{v^5} \left( 3 \log \frac{1-v}{1+v} + \frac{1}{1-v} - \frac{1}{1+v} \right).$$

It remains to make Taylor expansion in  $v$  and observe that the expression (A.12) involves only the even powers of  $v$ :  $v^{-2}$ ,  $v^0$ ,  $v^2$ ,  $v^4$ , ... ; the coefficient at  $v^{-2}$  is zero; the coefficients at all the rest powers of  $v$  are positive. As the result, the matrix

$$\Omega^+(v) = \|\Omega(\tau_j, \tau_{l+3})\|_{j,l=1,2,3} \quad (\text{A.13})$$

is positive definite and hence non-degenerate. Finally, the matrix

$$\|\Omega(\tau_j, \tau_l)\|_{j,l=1,\dots,6} = \begin{pmatrix} 0 & \Omega^+(v) \\ -\Omega^+(v) & 0 \end{pmatrix} \quad (\text{A.14})$$

is also non-degenerate.

## B Appendix: Decay of the Group of Free Wave Equation in Weighted Sobolev Spaces

We prove Lemma 18.2 for the wave equation written in the standard form

$$\dot{F}(t) = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} F(t), \quad F(t) = \begin{pmatrix} a(t) \\ e(t) \end{pmatrix}. \quad (\text{B.1})$$

For simplicity of notations let us denote the corresponding group  $W(t)$  instead of  $W^0(t)$ . Denote by  $W(x-y, t)$  the (distribution) integral matrix kernel of the operator  $W(t)$ . The Huygens principle for the group  $W(t)$  reads

$$W(x-y, t) = 0, \quad |x-y| > t. \quad (\text{B.2})$$

The energy conservation law for the group  $W(t)$  has the form

$$\int (|e(x, t)|^2 + |\nabla a(x, t)|^2) dx = \text{const}, \quad t \in \mathbb{R}.$$

In particular, this gives that

$$\|F(t)\|_{\mathcal{F}} \leq C \|F_0\|_{\mathcal{F}}, \quad t \in \mathbb{R}. \quad (\text{B.3})$$

The matrix kernel  $W(z, t)$  can be written explicitly since the solution to (B.1) has the form

$$a(\cdot, t) = \left[ \frac{\partial}{\partial t} R(t) * a_0 + R(t) * e_0 \right], \quad e(\cdot, t) = \dot{a}(\cdot, t). \quad (\text{B.4})$$

Here  $R(t) = R(\cdot, t)$ , and

$$R(x, t) = \frac{\delta(t - |x|)}{4\pi t}. \quad (\text{B.5})$$

*Step ii)* It remains to estimate  $\|W(t)F_0\|_{-\beta}$  for large  $t > 0$ . Let us fix an arbitrary  $t \geq 1$ , and split the initial function  $F_0$  in two terms,  $F_0 = F'_{0,t} + F''_{0,t}$  such that

$$\|F'_{0,t}\|_{\beta} + \|F''_{0,t}\|_{\beta} \leq C \|F_0\|_{\beta}, \quad t \geq 1, \quad (\text{B.6})$$

where  $C$  does not depend on  $t$ , and

$$F'_{0,t}(x) = 0, \quad |x| > t + 1, \quad (\text{B.7})$$

$$F''_{0,t}(x) = 0, \quad |x| < t. \quad (\text{B.8})$$

Now the estimate for  $W(t)F''_{0,t}$  follows by (B.3), (B.8) and (B.6) :

$$\begin{aligned} \|W(t)F''_{0,t}\|_{-\beta} &\leq \|W(t)F''_{0,t}\|_{\mathcal{F}} \leq C\|F''_{0,t}\|_{\mathcal{F}} \\ &\leq C_1\|F''_{0,t}\|_{\beta}(1+|t|)^{-\beta} \leq C_2\|F_0\|_{\beta}(1+|t|)^{-\beta}, \quad t \geq 1. \end{aligned} \quad (\text{B.9})$$

*Step iii)* It remains to estimate  $W(t)F'_{0,t}$ . By (B.4), (B.5) The support of  $e(\cdot, t), a(\cdot, t)$  is in the area  $\{|x| > t/2\}$ . Then

$$\|(1+|x|)^{-\beta}e(t)\|_{L^2} \leq C(1+t)^{-\beta}\|e(t)\|_{L^2} \leq C\|F_0\|_{L^2} \leq C\|F_0\|_{\beta}.$$

The norm  $\|(1+|x|)^{-\beta}\nabla a(t)\|_{L^2}$  can be estimated similarly.

## C Appendix: Computations for Invertibility of $A|_{\mathcal{Z}_{\beta,v}}$

For  $j = 1, 2, 3$  we have

$$\begin{aligned} \Omega(Y, \tau_j) &= -\langle e, \partial_j A_v \rangle + \langle a, \partial_j E_v \rangle - \langle a, \rho \rangle \cdot e_j = \\ &= -\int dk \hat{e} \frac{-ik_j(-\hat{\rho})}{D} \left( \frac{kv}{k^2} k - v \right) + \int dk \hat{a} \frac{-ik_j i(kv)\hat{\rho}}{D} \left( \frac{kv}{k^2} k - v \right) - \langle a, \rho \rangle \cdot e_j. \end{aligned}$$

Since  $\hat{e} \perp k$  and  $\hat{a} \perp k$ , the expression simplifies to

$$-i \int dk \hat{e} \frac{k_j \bar{\hat{\rho}}}{D} v - \int dk \hat{a} \frac{k_j (kv) \bar{\hat{\rho}}}{D} v - \int dk \hat{a} \bar{\hat{\rho}} \cdot e_j.$$

We substitute

$$\hat{e} = \frac{(kv)(kr)\hat{\rho}}{D} \left( v - \frac{kv}{k^2} k \right), \quad \hat{a} = \frac{i(kr)\hat{\rho}}{D} \left( v - \frac{kv}{k^2} k \right)$$

and obtain that the expression equals zero, since every integrand function is odd in  $k$ .

Further, we take the relations  $\hat{e} \perp k, \hat{a} \perp k$  into account and compute

$$\begin{aligned} \Omega(Y, \tau_{j+3}) &= \langle e, \partial_{v_j} A_v \rangle - \langle a, \partial_{v_j} E_v \rangle + r \cdot \partial_{v_j} P_v = \\ &= \int dk \hat{e} \frac{\bar{\hat{\rho}}}{D} \left( \frac{2k_j(kv)}{\hat{D}} v + e_j \right) - i \int dk \hat{a} \frac{\bar{\hat{\rho}}}{D} \left( \frac{k_j(k^2 + (kv)^2)}{\hat{D}} v + (kv)e_j \right) + r \cdot \partial_{v_j} P_v. \end{aligned}$$

We substitute the expressions for  $\hat{e}$  and  $\hat{a}$  and obtain:

$$\begin{aligned} \Omega(Y, \tau_{j+3}) &= r \cdot B_v^{-1} e_j + \int \frac{|\hat{\rho}|^2 dk}{D} r_j \\ &+ v^2 \int \frac{|\hat{\rho}|^2 (kv)^2 dk}{D^3} (kr) k_j - \int \frac{|\hat{\rho}|^2 (kv)^4 dk}{k^2 D^3} (kr) k_j \\ &+ \int \frac{|\hat{\rho}|^2 (kv)^2 dk}{D^3} (kr) k_j - v^2 \int \frac{|\hat{\rho}|^2 k^2 dk}{D^3} (kr) k_j - 3 \int \frac{|\hat{\rho}|^2 (kv)^2 dk}{k^2 D^2} (kr) k_j \\ &- \int \frac{|\hat{\rho}|^2 dk}{D^2} (kr) k_j + 2 \int \frac{|\hat{\rho}|^2 (kv) dk}{D^2} (kr) v_j + 2 \int \frac{|\hat{\rho}|^2 (kv) dk}{D^2} (rv) k_j. \end{aligned}$$

In the first line there is the  $j$ -th component of  $M_1 r$ , where  $M_1$  is a positive matrix. The second line equals

$$\int \frac{|\hat{\rho}|^2 (kv)^2 (k^2 v^2 - (kv)^2) dk}{k^2 D^3} (kr) k_j.$$

Note that  $k^2 v^2 - (kv)^2 \geq 0$ . Thus, one applies the nonnegative matrix  $k \otimes k$  integrated with a nonnegative weight to  $r$  and takes the  $j$ -th component. To all of the rest integrals we apply the change of variables

described in Appendix A. We make this change, omit the prime, and keep only the terms that lead to nonzero integrals. Thus,

$$\begin{aligned}
(kr)k_j &\mapsto (Vk \cdot r)(Vk)_j = \left[ \left( \sum_{l=1}^3 V_{1l}k_l \right) r_1 + \left( \sum_{l=1}^3 V_{2l}k_l \right) r_2 + \left( \sum_{l=1}^3 V_{3l}k_l \right) r_3 \right] \sum_{l=1}^3 V_{jl}k_l \mapsto \\
&\left( \sum_{l=1}^3 V_{1l}V_{jl}k_l^2 \right) r_1 + \left( \sum_{l=1}^3 V_{2l}V_{jl}k_l^2 \right) r_2 + \left( \sum_{l=1}^3 V_{3l}V_{jl}k_l^2 \right) r_3 \mapsto \\
&(V_{11}V_{j1}k_1^2 + (V_{12}V_{j2} + V_{13}V_{j3} + V_{11}V_{j1})k_2^2 - V_{11}V_{j1}k_2^2)r_1 + \dots = \\
&\frac{v_1v_j}{v^2}(k_1^2 - k_2^2)r_1 + \frac{v_2v_j}{v^2}(k_1^2 - k_2^2)r_2 + \frac{v_3v_j}{v^2}(k_1^2 - k_2^2)r_3 = \frac{k_1^2 - k_2^2}{v^2}((v \otimes v)r)_j.
\end{aligned}$$

Further,

$$k_1^2 - k_2^2 \mapsto k_1^2 - \frac{k_2^2 + k_3^2}{2} = k_1^2 - \frac{k_2^2 + k_3^2 + k_1^2 - k_1^2}{2} = \frac{3k_1^2 - k^2}{2}$$

and finally

$$(kr)k_j \mapsto \frac{3k_1^2 - k^2}{2}((v \otimes v)r)_j.$$

Then

$$\begin{aligned}
\int \frac{|\hat{\rho}|^2(kv)^2 dk}{D^3} (kr)k_j &\mapsto \frac{1}{2} \int \frac{|\hat{\rho}|^2 k_1^2 (3k_1^2 - k^2) dk}{D^3} ((v \otimes v)r)_j, \\
v^2 \int \frac{|\hat{\rho}|^2 k^2 dk}{D^3} (kr)k_j &\mapsto \frac{1}{2} \int \frac{|\hat{\rho}|^2 k^2 (3k_1^2 - k^2) dk}{D^3} ((v \otimes v)r)_j, \\
3 \int \frac{|\hat{\rho}|^2(kv)^2 dk}{k^2 D^2} (kr)k_j &\mapsto \frac{3}{2} \int \frac{|\hat{\rho}|^2 k_1^2 (3k_1^2 - k^2) dk}{k^2 D^2} ((v \otimes v)r)_j, \\
\int \frac{|\hat{\rho}|^2 dk}{D^2} (kr)k_j &\mapsto \frac{1}{2v^2} \int \frac{|\hat{\rho}|^2 (3k_1^2 - k^2) dk}{D^2} ((v \otimes v)r)_j.
\end{aligned}$$

Further, by the similar argument

$$(kr)v_j \mapsto \frac{k_1}{|v|}((v \otimes v)r)_j \text{ and } (rv)k_j \mapsto \frac{k_1}{|v|}((v \otimes v)r)_j.$$

Then

$$\begin{aligned}
2 \int \frac{|\hat{\rho}|^2(kv) dk}{D^2} (kr)v_j &\mapsto 2 \int \frac{|\hat{\rho}|^2 k_1^2 dk}{D^2} ((v \otimes v)r)_j, \\
2 \int \frac{|\hat{\rho}|^2(kv) dk}{D^2} (rv)k_j &\mapsto 2 \int \frac{|\hat{\rho}|^2 k_1^2 dk}{D^2} ((v \otimes v)r)_j.
\end{aligned}$$

In total we obtain  $((v \otimes v)r)_j$  multiplied by the following expression:

$$\left( \frac{11}{2} - \frac{3}{2v^2} \right) I_1 + \frac{1}{2v^2} I_2 - 2I_3 + \frac{3}{2} I_4 - \frac{9}{2} I_5 + \frac{1}{2} I_6, \quad (\text{C.1})$$

where the integrals  $I_1$  to  $I_5$  are introduced in Appendix A, and

$$I_6 = \int \frac{|\hat{\rho}|^2 k^4 dk}{D^3} = \frac{\pi R}{4v} \left( 3 \log \frac{1+v}{1-v} + \frac{6v}{1-v^2} + \frac{1}{(1-v)^2} - \frac{1}{(1+v)^2} \right).$$

It remains, similar to Appendix A, to make Taylor expansion in  $v$  and observe that the expression (C.1) involves only the even powers of  $v$ :  $v^{-2}$ ,  $v^0$ ,  $v^2$ ,  $v^4$ , ... ; the coefficient at  $v^{-2}$  is zero; the coefficients at all the rest powers of  $v$  are positive. As the result, the orthogonality conditions transform to  $Mr = 0$ , where  $M$  is a positive and hence non-degenerate matrix. This implies  $r = 0$ .



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